Planar maps, random walks and the circle packing theorem

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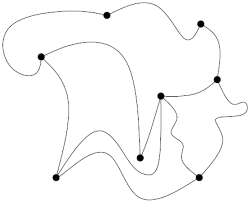
Basic terminology

- Planar map: a graph embedded in $\mathbb{R}^2$ so that vertices are mapped to points and edges to non-intersecting curves.



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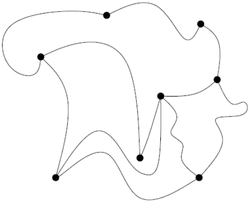

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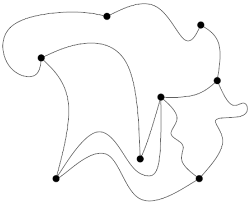

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- The simple random walk on a graph starts at an arbitrary vertex and in each step moves to a uniformly chosen neighbor.

Circle packing

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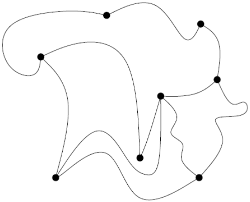

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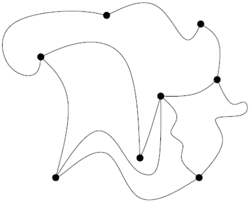

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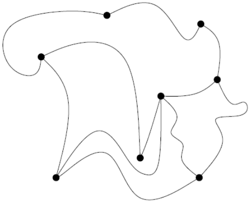

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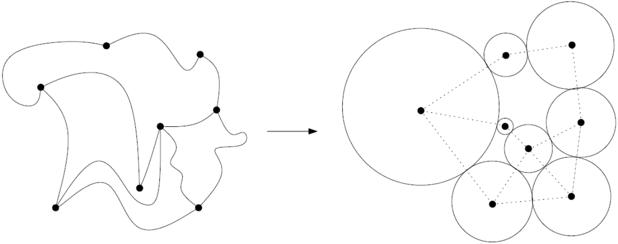

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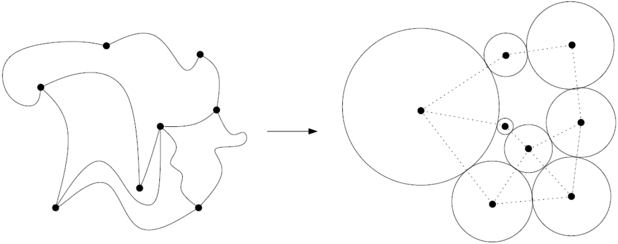

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If G is a triangulation, then the drawing is unique up to Möbius transformations and reflections.

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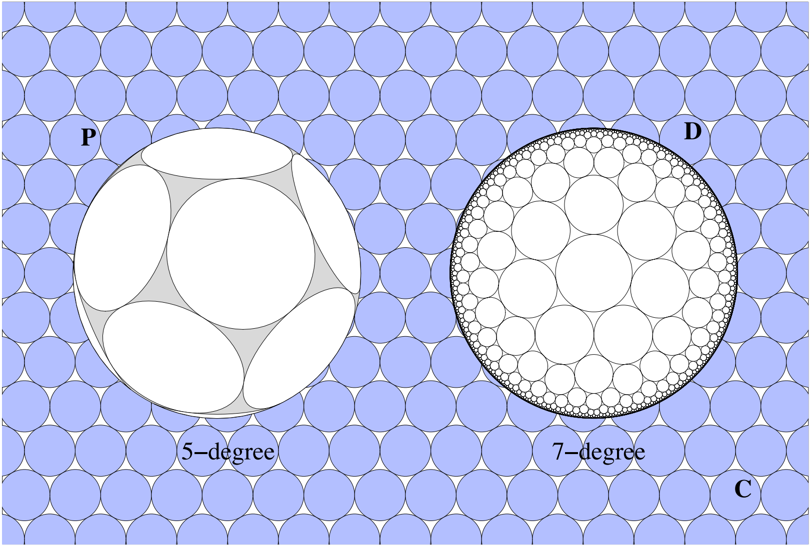
- The set of accumulation points $A(P)$ is the boundary of the carrier $\mathrm{carr}(P)$.



Picture due to Ken Stephenson.

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- We call a circle packing of an infinite triangulation a packing **in the disc** if its carrier is the unit disc $\mathbb{D}$, and **in the plane** if its carrier is $\mathbb{C}$.

Theorem (He-Schramm '95)

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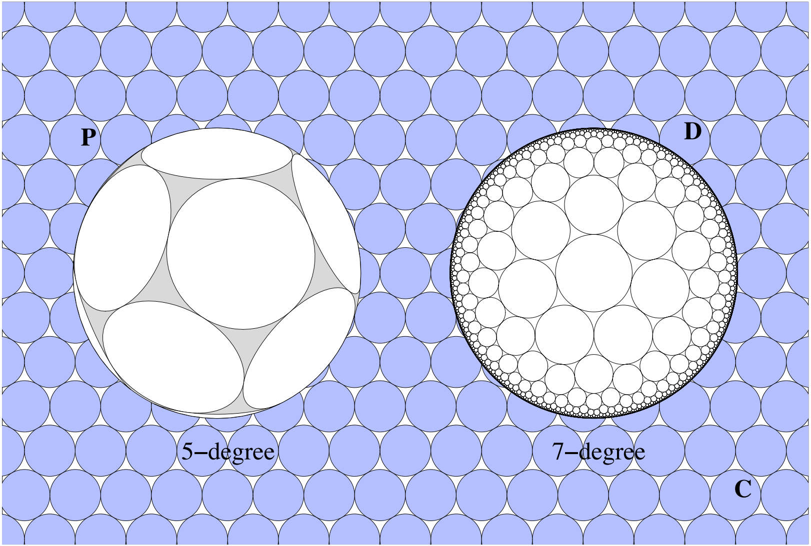
Theorem (Schramm's rigidity '91)

The above circle packing is unique up to Möbius transformations of the plane or the sphere as appropriate.



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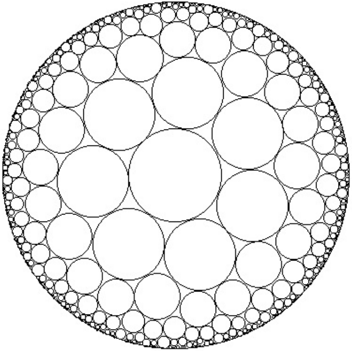
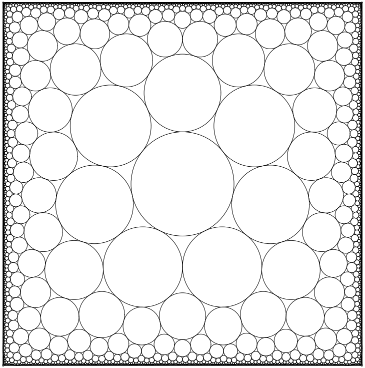
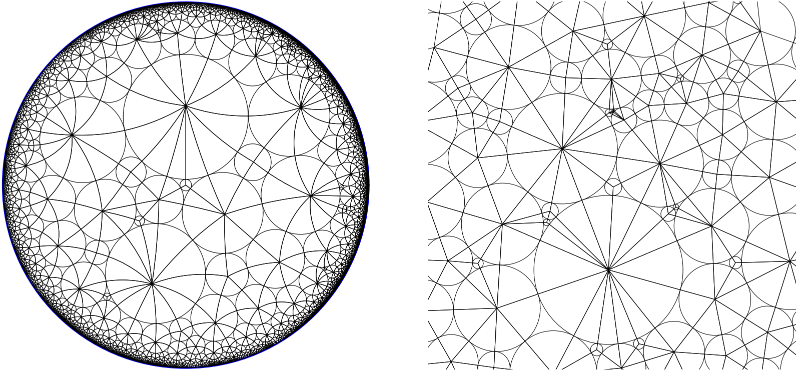
7-regular hyperbolic tessellation





Circle packing also gives us a drawing of the graph with either straight lines or hyperbolic geodesics depending on the type.



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Theorem (Benjamini-Schramm '96)

Assume G is a bounded degree, CP hyperbolic triangulation of the plane circle packed in the unit disc $\mathbb{D}$. Then the random walk converges to $\partial\mathbb{D}$, the exit measure is non-atomic and has full measure.

A dichotomy for bounded degree plane triangulations

If G is bounded degrees triangulation of the plane, then either:

> *Random walk on G is recurrent, G is CP parabolic and all bounded harmonic functions are constant,*

or

> *Random walk on G is transient, G is CP hyperbolic and any bounded Borel $g : \partial \mathbb{D} \to \mathbf{R}_+$ extends to G.*

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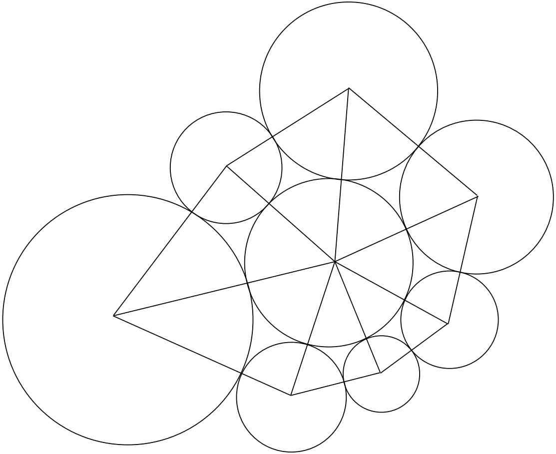
Theorem: There are no other bounded harmonic functions (Angel, Barlow, Gurel-Gurevich, N. '13).

The ring lemma (Rodin-Sullivan ’87)

Consider a circle packing of a bounded degree triangulation.
The ring lemma: The ratios of radii of tangent circles is bounded.



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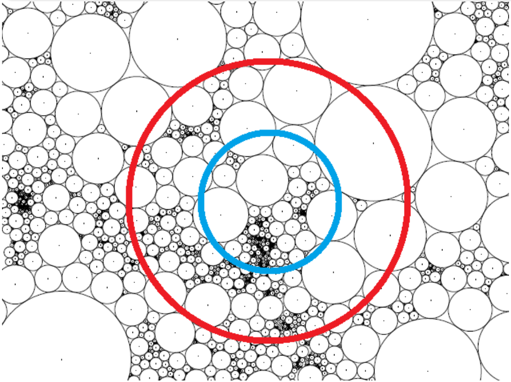
Lemma (Gurel-Gurevich, N. 2012)

The electric effective resistance across an annulus of fixed proportions is at least a constant.



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Hence, the sum is bounded. □

Proof 2 of He-Schramm Theorem (the adult proof)

Let G be a triangulation. Construct a Riemannian surface $\mathcal{M}(G)$ by gluing Euclidean equilateral triangles of edge length 1 according to the combinatorics of the graph. The surface is flat every everywhere except the vertices, on which the chart is $z \mapsto z^{6/\deg}$. The surface is equipped with the natural shortest (continuous) path metric.

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Corollary: The random walk on G is recurrent iff Brownian motion on $\mathcal{M}(G)$ is recurrent.

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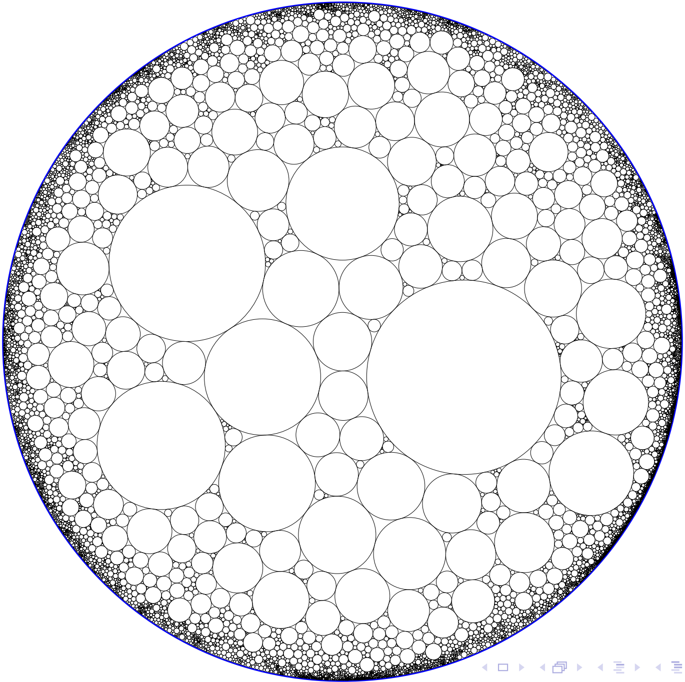
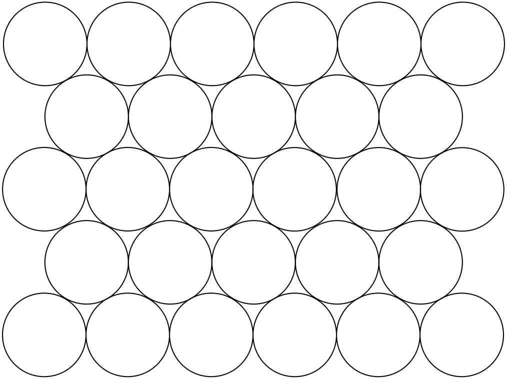
Stronger result: A bounded degree circle packing P is recurrent iff its set of accumulation points is missed by planar Brownian motion almost surely (Gurel-Gurevich, N., Souto 2014).



Without bounded degree, there is no theory!



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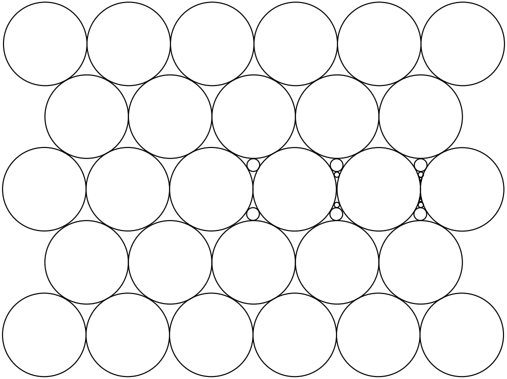

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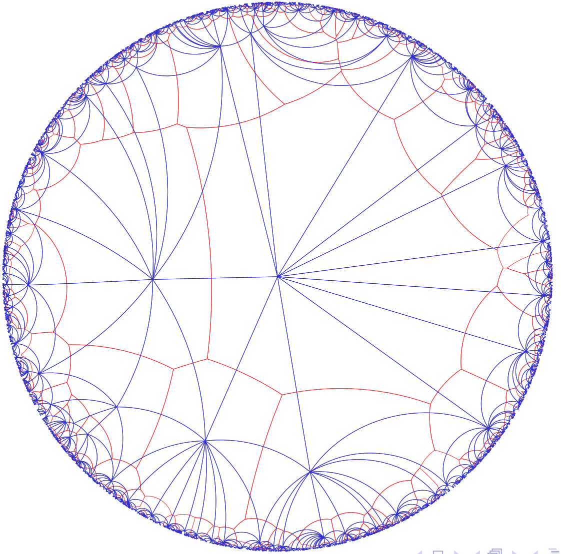
And, in the hyperbolic case,

- Question 3: Does the walker converge to a point in the boundary of the disc? Does the law of the limit have full support and no atoms almost surely?
- Question 4: Is the unit circle a realisation of the Poisson boundary?

Example 1: Poisson-Voronoi triangulation



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The G_n Benjamini-Schramm converge to a random rooted graph (G, ρ) if for each fixed r, the balls of radius r converge in distribution:

$$B_r(G_n, \rho_n) \xrightarrow{d} B_r(G, \rho)$$

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- Large tori $\mathbb{Z}^d/n\mathbb{Z}^d$ and large boxes $[-n,n]^d$ Benjamini-Schramm converge to the lattice $\mathbb{Z}^d$.

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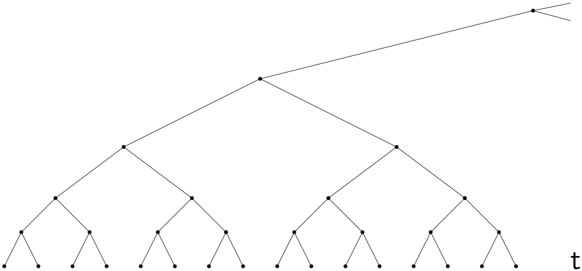

the **canopy tree**.

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Theorem (Angel and Schramm '03)

Let T_n be a uniformly random triangulation of the sphere. The Benjamini-Schramm limit of T_n as $n \to \infty$ exists. We call this limit the UIPT – Uniform Infinite Plane Triangulation.

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Theorem (Gurel-Gurevich, N. 2013)

The UIPT is almost surely recurrent.

Example 2: hyperbolic triangulations with Markov property

The UIPT has a natural Markov property – if we explore the UIPT, revealing more of it by peeling away at the boundary, the law of the part we haven't uncovered yet depends only on the length of the boundary of the piece we've already revealed.

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Are there any other triangulations with this property?

Theorem (Angel and Ray '13, Curien '13)

Yes. The laws Markovian plane triangulations form a one-parameter parameter family T_κ, $\kappa \in (0, 2/27]$. *The endpoint* $\kappa = 2/27$ *is the UIPT, all the others have 'hyperbolic flavour'.*

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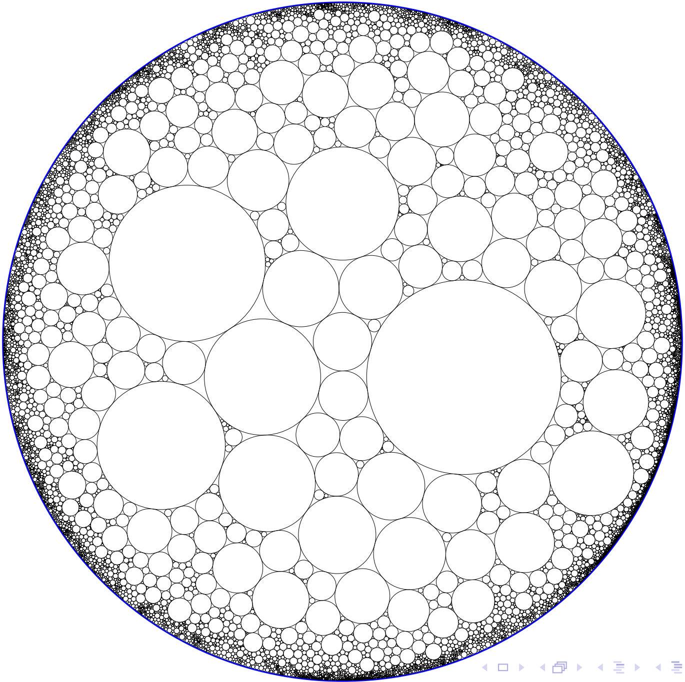
Conjecturally, the hyperbolic triangulations are Benjamini-Schramm limits of uniform triangulations with *n* vertices of surfaces of genus *cn*.



Unimodular random triangulations

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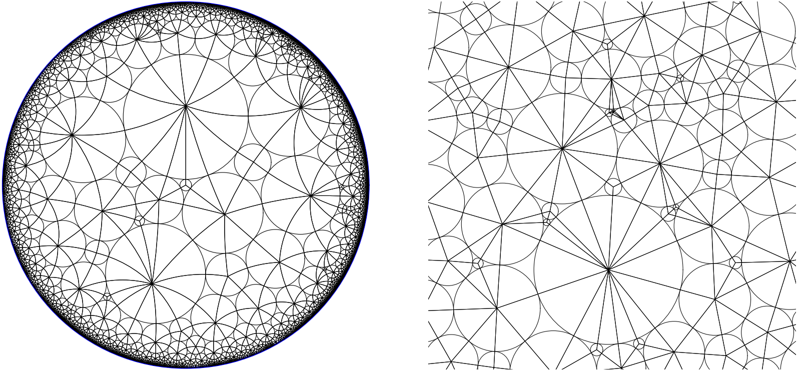
Corollary

Theorem (Benjamini, Schramm 1996)

Any distributional limit of finite planar triangulations is CP-parabolic. Hence, if the degrees are bounded, the resulting graph is almost surely recurrent for the simple random walk.

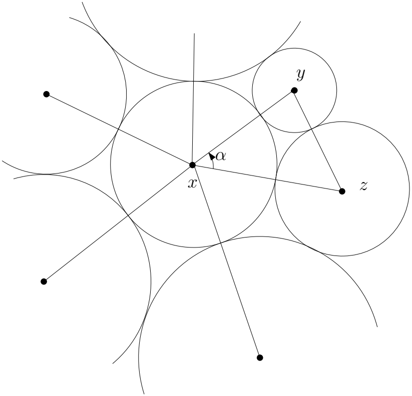
Our proof avoids the use of a powerful, yet rather technical, lemma of Benjamini-Schramm known as the “magical lemma”.

Circle packing also gives us a drawing of the graph with either straight lines or hyperbolic geodesics depending on the type



Proof: CP type and average degree (parabolic case)

For each corner, send α from x to each of x, y, z.



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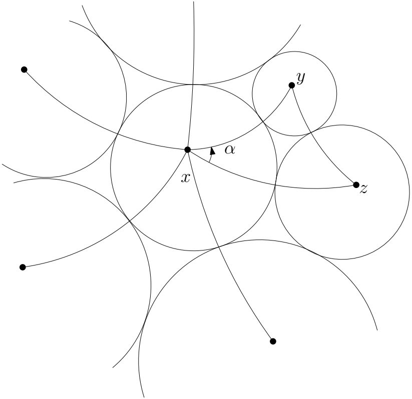

Mass out is 6π. Mass in is less than $\pi \deg(x)$.

Non-amenability

- Recall that the (edge) Cheeger constant of an infinite graph G is defined to be

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- Are unimodular CP hyperbolic triangulations non-amenable? No, the condition is too strong.

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- Easy fact: $\iota^{\text{inv}}(G) > 0 = \mathbf{E}\deg(\rho) - \alpha(G)$ where

$$\alpha(G) = \sup \left\{ \mathbf{E}[\deg_\omega(\rho)] : \omega \text{ a finite percolation} \right\}.$$

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- Critical Galton-Watson tree conditioned to survive is invariantly amenable.
- Fact (Aldous-Lyons '07): If (G, ρ) is unimodular and is a.s. recurrent, then it is invariantly amenable.

CP Hyperbolic triangulations

By Euler's formula the average degree of any *finite* planar triangulation is at most 6. Hence,

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Boundary theory

Theorem (Angel, Hutchcroft, N., Ray '14)

Let (G, ρ) be a CP hyperbolic unimodular random planar triangulation with $\mathbb{E}[\deg^2(\rho)] < \infty$ and let $\mathcal{C}$ be a circle packing of G in the unit disc. The following hold conditional on (G, ρ) almost surely:

1. *The random walk almost surely has $X_n \to X_\infty \in \partial\mathbb{D}$*
2. *The law of X_∞ has full support and no atoms.*
3. *$\partial\mathbb{D}$ is a realisation of the Poisson-Furstenberg boundary of G.*

Proof of convergence: $X_n \to X_\infty \in \partial\mathbb{D}$

- Assume G is **really** non-amenable and has degrees bounded by M. Then for some $a < 1$ and any v

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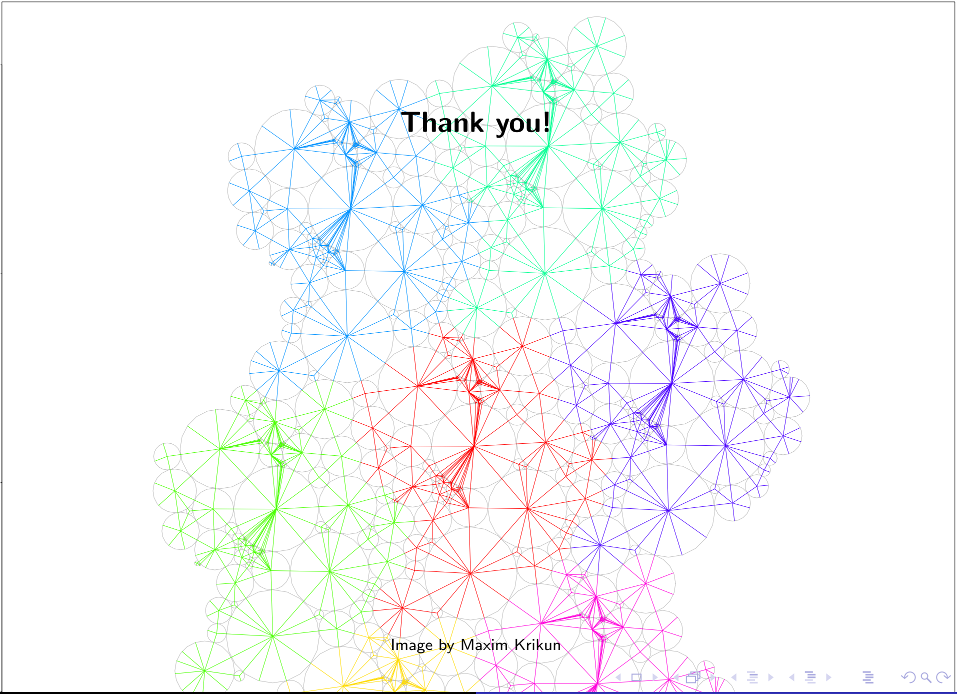
- When G is only invariantly non-amenable, perform the same argument on the "dense" non-amenable subgraph and argue that in the times the random walker is not in this subgraph things cannot go very badly.



Thank you!

Image by Maxim Krikun