# Regularity for almost minimizers with free boundary

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# Minimizers with free boundary

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, $q_\pm \in L^\infty(\Omega)$ and

$$K(\Omega) = \left\{u \in L^1_{loc}(\Omega)\,;\ \nabla u \in L^2(\Omega)\right\}.$$

**Minimizing problem with free boundary:** Given $u_0 \in K(\Omega)$ minimize

$$J(u) = \int_\Omega |\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x)$$

among all $u = u_0$ on $\partial\Omega$.

- One phase problem arises when $q_- \equiv 0$ and $u_0 \geq 0$.
- The general problem is know as the two phase problem.

# Alt-Caffarelli

- Minimizers for the one phase problem exist.
- If $u$ is a minimizer of the one phase problem, then $u \geq 0$, $u$ is subharmonic in $\Omega$ and

$$\Delta u = 0 \text{ in } \{u > 0\}$$

- $u$ is locally Lipschitz in $\Omega$.
- If $q_+$ is bounded below away from 0, that is there exists $c_0 > 0$, such that $q_+ \geq c_0$, then:
  - for $x \in \{u > 0\}$

$$\frac{u(x)}{\delta(x)} \sim 1 \text{ where } \delta(x) = \text{dist}(x, \partial\{u > 0\})$$

  - $\{u > 0\} \cap \Omega$ is a set of locally finite perimeter, thus $\partial\{u > 0\} \cap \Omega$ is (n-1)-rectifiable.

# Alt-Caffarelli-Friedman

- Minimizers for the two phase problem exist.
- If $u$ is a minimizer of the two phase problem, then $u^{\pm}$ are subharmonic and

$$\Delta u = 0 \text{ in } \{u > 0\} \cup \{u < 0\}$$

- $u$ is locally Lipschitz in $\Omega$.
- If $q_{\pm}$ are bounded below away from 0, then
  - for $x \in \{u^{\pm} > 0\}$

$$\frac{u^{\pm}(x)}{\delta(x)} \sim 1 \text{ where } \delta(x) = \text{dist}(x, \partial\{u^{\pm} > 0\})$$

  - $\{u^{\pm} > 0\} \cap \Omega$ are sets of locally finite perimeter.

# Regularity of the free boundary $\Gamma(u)$

- If $u$ is a minimizer for the one phase problem $\Gamma(u) = \partial\{u > 0\}$
- If $q_+$ is Hölder continuous and $q_+ \geq c_0 > 0$ then
  - if $n = 2, 3$, $\Gamma(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold.
  - if $n \geq 4$, $\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u)$ where $\mathcal{R}(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold and $\mathcal{S}(u)$ is a closed set of Hausdorff dimension less than n-3.

- If $u$ is a minimizer for the two phase problem $\Gamma(u) = \partial\{u > 0\} \cup \partial\{u < 0\}$
- If $q_\pm$ are Hölder continuous $q_+ > q_- \geq 0$ and $q_+ \geq c_0 > 0$ then
  - if $n = 2, 3$, $\Gamma(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold.
  - if $n \geq 4$, $\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u)$ where $\mathcal{R}(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold and $\mathcal{S}(u)$ is a closed set of Hausdorff dimension less than n-3.

# Contributions

- One phase problem:
  - n=2, Alt-Caffarelli
  - $n \geq 3$, Alt-Caffarelli, Caffarelli-Jerison-Kenig / Weiss
- Two phase problem:
  - n=2 , Alt-Caffarelli-Friedman
  - $n \geq 3$, Alt-Caffarelli-Friedman, Caffarelli-Jerison-Kenig / Weiss
- DeSilva-Jerison: There exists a non-smooth minimizer for $J$ in $\mathbb{R}^7$ such that $\Gamma(u)$ is a cone.

# Almost minimizers for the one phase problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, $q_+ \in L^\infty(\Omega)$ and

$$K_+(\Omega) = \left\{ u \in L^1_{loc}(\Omega)\,;\, u \geq 0 \text{ } a.e. \text{ in } \Omega \text{ and } \nabla u \in L^2_{loc}(\Omega) \right\}$$

- $u \in K_+(\Omega)$ is a $(\kappa, \alpha)$-almost minimizers for $J^+$ in $\Omega$ if for any ball $B(x,r) \subset \Omega$

$$J^+_{x,r}(u) \leq (1 + \kappa r^\alpha) J^+_{x,r}(v)$$

for all $v \in K_+(\Omega)$ with $u = v$ on $\partial B(x,r)$, where

$$J^+_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2 \, \chi_{\{v>0\}}.$$

# Almost minimizers for the two phase problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, $q_\pm \in L^\infty(\Omega)$ and

$$K(\Omega) = \left\{ u \in L^1_{loc}(\Omega)\,;\ \nabla u \in L^2_{loc}(\Omega) \right\}.$$

- $u \in K(\Omega)$ is a $(\kappa, \alpha)$-almost minimizers for $J$ in $\Omega$ if for any ball $B(x,r) \subset \Omega$

  $$J_{x,r}(u) \leq (1 + \kappa r^\alpha) J_{x,r}(v)$$

  for all $v \in K(\Omega)$ with $u = v$ on $\partial B(x,r)$, where

  $$J_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2\, \chi_{\{v>0\}} + q_-^2\, \chi_{\{v>0\}}.$$

# Almost minimizers are continuous

**Theorem:** Almost minimizers of $J$ are continuous in $\Omega$. Moreover if $u$ is an almost minimizer for $J$ there exists a constant $C > 0$ such that if $B(x_0, 2r_0) \subset \Omega$ then for $x, y \in B(x_0, r_0)$

$$|u(x) - u(y)| \leq C|x - y| \left(1 + \log \frac{2r_0}{|x - y|}\right).$$

**Remark:** Since almost-minimizers do not satisfy an equation, good comparison functions are needed.

# Sketch of the proof

To prove regularity of $u$, an almost minimizer for $J$, we need to control the quantity

$$\omega(x,s) = \left( \fint_{B(x,s)} |\nabla u|^2 \right)^{1/2}$$

for $s \in (0,r)$ and $B(x,r) \subset \Omega$.

Consider $u_r^*$ satisfying $\Delta u_r^* = 0$ in $B(x,r)$ and $u_r^* = u$ on $\partial B(x,r)$. Then since $|\nabla u_r^*|^2$ is subharmonic

$$\begin{aligned} \omega(x,s) &\leq \left( \fint_{B(x,s)} |\nabla u - \nabla u_r^*|^2 \right)^{1/2} + \left( \fint_{B(x,s)} |\nabla u_r^*|^2 \right)^{1/2} \\ &\leq \left( \frac{r}{s} \right)^{n/2} \left( \fint_{B(x,r)} |\nabla u - \nabla u_r^*|^2 \right)^{1/2} + \left( \fint_{B(x,r)} |\nabla u_r^*|^2 \right)^{1/2} \end{aligned}$$

# The almost minimizing property comes in

Since $\Delta u_r^* = 0$ in $B(x,r)$ and $u_r^* = u$ on $\partial B(x,r)$ and $q_\pm \in L^\infty(\Omega)$

$$
\begin{aligned}
\int_{B(x,r)} |\nabla u - \nabla u_r^*|^2 &= \int_{B(x,r)} |\nabla u|^2 - \int_{B(x,r)} |\nabla u_r^*|^2 \\
&\leq (1+\kappa r^\alpha) \int_{B(x,r)} |\nabla u_r^*|^2 - \int_{B(x,r)} |\nabla u_r^*|^2 + C r^n \\
&\leq \kappa r^\alpha \int_{B(x,r)} |\nabla u_r^*|^2 + C r^n \\
&\leq \kappa r^\alpha \int_{B(x,r)} |\nabla u|^2 + C r^n.
\end{aligned}
$$

# Iteration scheme

$$\omega(x,s) \quad \leq \quad \left(1 + C\left(\frac{r}{s}\right)^{n/2} r^{\alpha/2}\right)\omega(x,r) + C\left(\frac{r}{s}\right)^{n/2}.$$

Set $r_j = 2^{-j}r$ for $j \geq 0$, iteration yields

$$\omega(x, r_j) \leq C\omega(x,r) + Cj,$$

which for $s \in (0, r)$ ensures

$$\omega(x,s) \leq C\left(\omega(x,r) + \log\frac{r}{s}\right).$$

# Local regularity on each phase

**Theorem:** Let $u$ be an almost minimizer for $J$ in $\Omega$. Then $u$ is locally Lipschitz in $\{u > 0\}$ and in $\{u < 0\}$.

**Theorem:** Let $u$ be an almost minimizer for $J$ in $\Omega$. Then there exists $\beta \in (0, 1)$ such that $u$ is $C^{1,\beta}$ locally in $\{u > 0\}$ and $\{u < 0\}$.

**Proof:** Refine the argument above.

# Local regularity for minimizers

**Theorem** [AC], [ACF]: Let $u$ be a minimizer for $J$ in $\Omega$. Then $u$ is locally Lipschitz.

**Elements of the proof:**

- $u^{\pm}$ are subharmonic in $\Omega$,
- $u$ harmonic on $\{u^{\pm} > 0\}$,
- the 2-phase case requires a monotonicity formula introduced by Alt-Caffarelli-Friedman [ACF], that is

$$\Phi(r) = \frac{1}{r^4}\left(\int_{B(x,r)} \frac{|\nabla u^+|^2}{|x-y|^{n-2}}\,dy\right)\left(\int_{B(x,r)} \frac{|\nabla u^-|^2}{|x-y|^{n-2}}\,dy\right)$$

  is an increasing function of $r > 0$.

# Local regularity for almost minimizers

**Theorem:** Let $u$ be an almost minimizer for $J$ in $\Omega$. Then $u$ is locally Lipschitz.

**Elements of the proof:**

- analysis of the interplay between

$$m(x,r) = \frac{1}{r}\fint_{\partial B(x,r)} u, \ \ \frac{1}{r}\fint_{\partial B(x,r)} |u| \ \text{ and } \ \omega(x,r) = \left(\fint_{B(x,s)} |\nabla u|^2\right)^{1/2}$$

- the 2-phase case requires an almost [ACF]-monotonicity formula, i.e. we need to control the oscillation of $\Phi(r)$ on small intervals.

# Sketch of the proof

For $1 \ll K$ and $0 < \gamma \ll 1$ if $B(x, 2r) \subset \Omega$ consider:

- **Case 1:**

$$\begin{cases} \omega(x,r) & \geq & K \\ |m(x,r)| & \geq & \gamma\,(1+\omega(x,r)) \end{cases}$$

- **Case 2:**

$$\begin{cases} \omega(x,r) & \geq & K \\ |m(x,r)| & < & \gamma\,(1+\omega(x,r)) \end{cases}$$

- **Case 3:**

$$\omega(x,r) \leq K$$

# Case 1

If $u$ is an almost minimizer for $J$ in $\Omega$, $B(x, 2r) \subset \Omega$ and

$$\begin{cases} \omega(x,r) & \geq & K \\ |m(x,r)| & \geq & \gamma\,(1+\omega(x,r)) \end{cases}$$

then there exists $\theta \in (0,1)$ such that $u \in C^{1,\beta}(B(x,\theta r))$ and

$$\sup_{B(x,\theta r)} |\nabla u| \lesssim \omega(x,r).$$

# Cases 2 & 3

If $u$ is an almost minimizer for $J^+$ in $\Omega$, $B(x, 2r) \subset \Omega$ and

$$\begin{cases} \omega(x,r) & \geq & K \\ m(x,r) & < & \gamma\,(1+\omega(x,r)) \end{cases}$$

then for $\theta \in (0,1)$ there exists $\beta \in (0,1)$ such that

$$\omega(x, \theta r) \leq \beta \omega(x, r).$$

If only cases 2 and 3 occur then

$$\limsup_{s \to 0} \omega(x, s) \lesssim K$$

and if $x$ is a Lebesgue point of $\nabla u$ then

$$|\nabla u(x)| \lesssim K.$$

# Remarks

If $u$ is an almost minimizer for $J$, Case 2 requires understanding the relationship between

$$|m(x,r)| = \left| \frac{1}{r} \fint_{\partial B(x,r)} u \right| \quad \text{and} \quad \frac{1}{r} \fint_{\partial B(x,r)} |u|.$$

**Almost monotonicity formula:**
Let $u$ be an almost minimizer for $J$ in $\Omega$. There exists $\delta > 0$ so that for $K \Subset \Omega$ there are constants $r_K > 0$ and $C_K > 0$ such that for $x \in \Gamma(u) \cap K$ and $0 < s < r < r_K$

$$\Phi(s) \leq \Phi(r) + C_K r^\delta,$$

# Understanding the free boundary for almost minimizers: non-degeneracy

Let $u$ be an almost minimizers for $J^+$ in $\Omega$ with $q_+ \in L^\infty(\Omega) \cap C(\Omega)$. Let

$$\Gamma(u) = \partial\{u > 0\}.$$

Assume

$$q_+ \geq c_0 > 0,$$

then there exists $\eta > 0$ so that if $x_0 \in \Gamma(u)$ and $B(x_0, 2r_0) \subset \Omega$ then for $r \in (0, r_0)$

$$\frac{1}{r} \fint_{\partial B(x_0,r)} u^+ \geq \eta$$

and

$$u(x) \geq \frac{\eta}{4}\,\delta(x) \quad \text{for} \quad x \in B(x_0, r_0) \cap \{u > 0\}.$$

# Structure of $\Gamma(u)$

Let $u$ be an almost minimizers for $J^+$ in $\Omega \subset \mathbb{R}^n$ with $q_+ \in L^\infty(\Omega) \cap C(\Omega)$ such that $q_+ \geq c_0 > 0$. Then

- $\{u > 0\} \subset \Omega$ is "locally" NTA.
- For $x_0 \in \Gamma(u)$ with $B(x_0, 2r_0) \subset \Omega$ there exists an Ahlfors regular measure $\mu_0$ supported on $B(x_0, r_0) \cap \Gamma(u)$.
- $\Gamma(u)$ is $(n-1)$-uniformly rectifiable.
- $\{u > 0\} \cap \Omega$ is a set of locally finite perimeter.

# Related questions

- Under the assumptions that $q_+ \in L^\infty(\Omega) \cap C^\gamma(\Omega)$ and $q_+ \geq c_0 > 0$, we expect that, for $u$ almost minimizer of $J^+$ in $\Omega$,

$$\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u)$$

  where $\mathcal{R}(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold and $\mathcal{S}(u)$ is a closed set of $(n-1)$-Hausdorff measure 0.

- We expect similar results for almost minimizers of functionals of the type:

$$J(u) = \int_\Omega \left( |\nabla u(x)|_g^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x) \right) dv_g,$$

  where $|\nabla u|_g$ denotes the norm of $\nabla u$ computed in the metric $g$ , $v_g$ is the corresponding volume and $g$ is assumed to be Hölder continuous.