MIT PRIMES CIRCLE 2025
DYNAMICAL SYSTEMS AND CHAOS THEORY

MARKO MANO, KALLE NIKULA-GILL, DEREK YIN

Abstract. The paper will explore the fundamental principles of dynamical systems and chaos theory - more precisely, discrete dynamical systems. The essential concept behind dynamical systems is the iteration of a function and the investigation of a function's chaotic or periodic behavior over time. Additionally, fixed points and periodic points will be discussed. The focal point of the study is the quadratic family, whose behavior is analyzed as the parameter c (a constant that vertically shifts the function) varies, leading to changes in the behavior of fixed points. Furthermore, we introduce the notion of attracting, repelling, and neutral fixed points through derivative criteria, and develop an understanding of bifurcations. We will explore quadratic functions in the real number plane and further discuss the existence of fixed points and their behavior (attracting, repelling, or neutral). The understanding of quadratic functions will be applied from the real number plane to the complex plane. The transition from real to complex dimensions allows the extension from 1D to 2D space through Julia sets and the Mandelbrot set. We will explore these sets, their construction, and some of their properties.

1. Introduction

A dynamical system refers to any system that is changing over time. Dynamical systems theory is a branch of mathematics that allows mathematicians to predict and model the evolution of a system (discrete and non-discrete). The study of dynamical systems possesses significant applications in finance, through interest accumulation over time, and ecology through the iteration of the logistic function to model the evolution of a population over time.

Definition 1.1. A dynamical system is fundamentally considered "chaotic" when small alterations in the initial conditions result in drastically diverse effects (known popularly as the "butterfly effect").

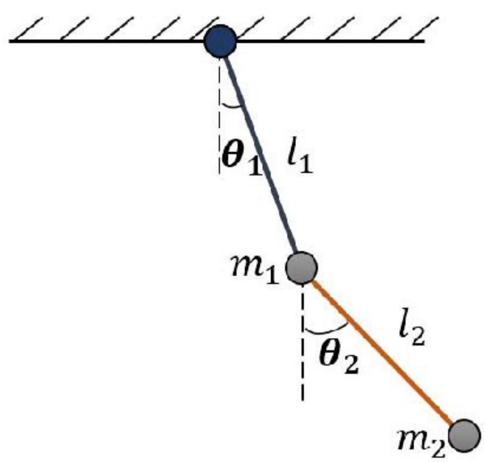


Figure 1. The double pendulum is a dynamical system that exhibits chaotic behavior.

2. Iterative Functions

Definition 2.1. A function is a relationship that assigns to each element x (from *Set X*) exactly one element of y (from *Set Y*).

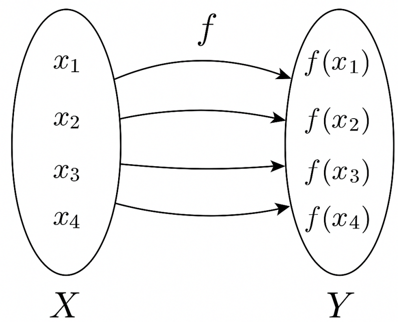


Figure 2. Basic Definition of Function

According to the basic notion of a function showcasing a relationship between an input and an output, the process of iterating functions consists of pushing an input into a function, returning an output, and computing the output as the new input to observe the function's behavior in relation to its prior value.

Definition 2.2. For a continuous function $f : X \to X$, the n-th iterate is defined by:

$$f^0(x) = x, \quad f^n(x) = f(f^{n-1}(x))$$

This defines a *discrete-time dynamical system.*

$$\begin{aligned} x_0 &= 0 \\ f(x_0) &= x_1 \\ f(x_1) &= x_2 = f(f(x_0)) \\ f(x_2) &= x_3 = f(f(f(x_0))) \end{aligned}$$

3. Functional Orbits and Cycles

One of the essential objectives of the study on dynamical systems is to understand point evolution through repeated use of a function. The repeated iteration of a function generates a sequence of values known as an *orbit*, which directly showcases the dynamics of a system over a period of time. The orbit of a function is directly observed and analyzed to determine key characteristics related to the equilibrium and divergence of the points that define the dynamics of a system. The recognizable reappearance of patterns, which can form cycles, periods, and fixed points, provides key insight into the chaotic behavior of the function.

We introduce in this section the exact definition of orbits and clarify what characteristics define them, laying the basis for a fuller discussion of periodicity and convergence within iterative processes.

Definition 3.1. In a dynamical system, an orbit is the sequence of points obtained by repeatedly applying a function to an initial point.

Mathematically, for a function $f : X \to X$, the **orbit** of a point $x_0 \in X$ is the set:

$$\mathcal{A}(x_0) = \{x_0, f(x_0), f(f(x_0)), f^3(x_0), \dots\}$$

Through the iteration of the function, the behavior of the function can be analyzed to make predictions.

3.1. **Example.** Let $g(x) = 2x + 5$ and $x_0 = 1$. Then the orbit of x_0 is:

$\mathcal{K}(1) = \{1,\ g(1) = 7,\ g^2(1) = g(7) = 19,\ g^3(1) = g(19) = 43,\ g^4(1) = g(43) = 91,\ g^5(1) = g(91) = 187,\ \dots\}$

Final Orbit Values for $g(x) = 2x + 5$ Starting at $x_0 = 1$:

$$\mathcal{K}(1) = \{1,\ 7,\ 19,\ 43,\ 91,\ 187\ \dots\}$$

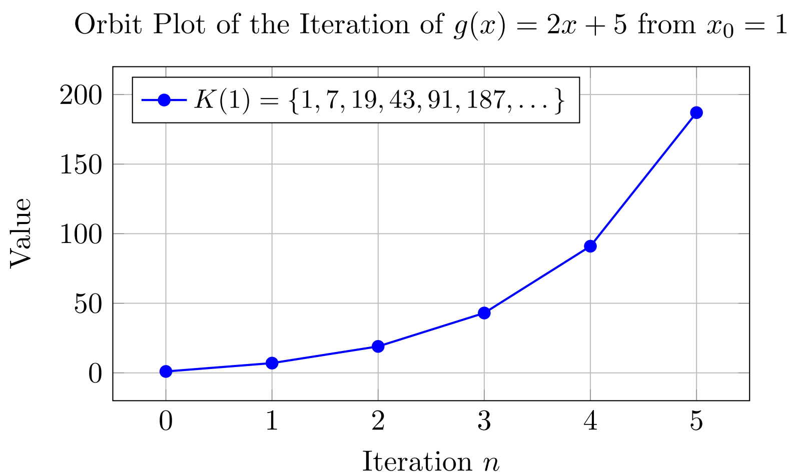


FIGURE 3. Orbit Plot of the Iteration of $g(x) = 2x + 5$ Starting at $x_0 = 1$

4. Fixed points and periodic points

In the study of discrete dynamical systems, it is fundamental to understand the long-term iterated behavior of functions. Two of the main notions capturing this behavior are **fixed points** and **periodic points**. Fixed points occur when an input does not change under the application of a function, while periodic points return to the same value after a finite number of iterations. These notions offer a starting point for analyzing the stability, convergence, and qualitative properties of dynamical systems. We will explicitly introduce the formal definition of fixed points, discuss their classification based on stability—namely attracting, repelling, or neutral—and introduce periodic points through illustrating examples and the dynamics of iterated functions.

Definition 4.1. A point $x \in \mathbb{R}$ is a **fixed point** of the function f if:

$$f(x) = x$$

The behavior of the iterative sequence $\{x_n\}$ near x depends on the derivative $f'(x)$:

- **Attracting**: If $|f'(x)| < 1$, then points near x converge to x. x is called a *locally attracting fixed point.*
- **Repelling**: If $|f'(x)| > 1$, then points near x diverge away from x. x is called a *locally repelling fixed point.*
- **Neutral**: If $|f'(x)| = 1$, then the fixed point is *neutral.*

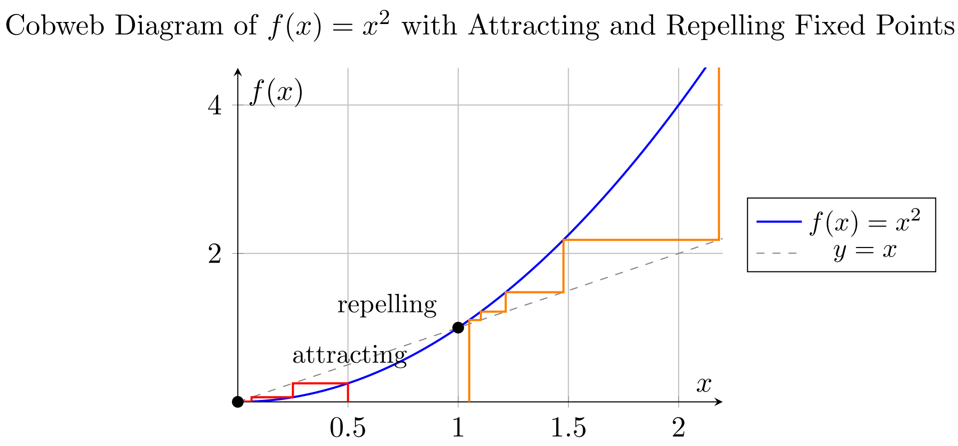


FIGURE 4. Cobweb Diagram for $f(x) = x^2$

4.1. **Example.** Let the function be defined as:

$$f(x) = 2 - x$$

We compute the iterates starting from $x_0 = 0$:

$$f(0) = 2, \quad f(2) = 0, \quad f(0) = 2, \quad f(2) = 0, \dots$$

Thus, the orbit of $x_0 = 0$ is:

$$\mathcal{A}(0) = \{0, 2, 0, 2, \dots\}$$

This orbit is considered a **2-cycle** or a **period 2**, because the iteration of the original function (f(x)=2-x) twice returns the original value:

$$f(f(x)) = f(2 - x) = 2 - (2 - x) = x$$

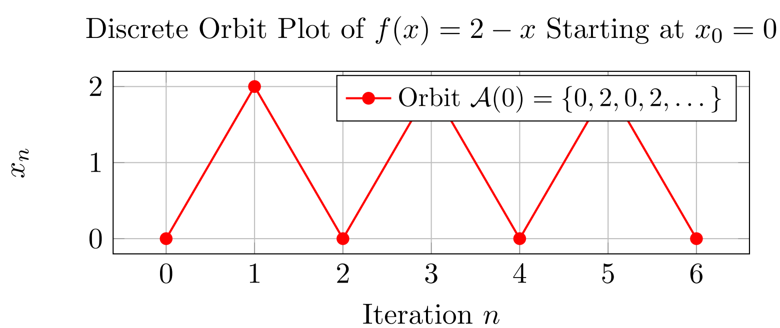


FIGURE 5. Time Series of the 2-Cycle for $f(x) = 2 - x$

4.2. **Example.** We say x_0 is a *fixed point* of f if $f(x_0) = x_0$. We say x_0 is *periodic with period* n with $f^n(x_0) = x_0$ but $f^k(x_0) \neq x_0$ for $1 \leq k < n$.

$$f(z) = z^2 - 1;$$
$$f(0) = -1$$
$$f(-1) = 0$$

so $\{0, -1\}$ form a 2-cycle.

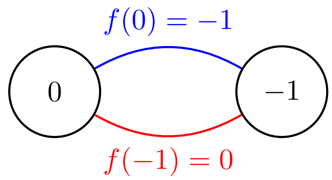


FIGURE 6. Directed Graph of the 2-cycle for $f(z) = z^2 - 1$

5. Bifurcations

Definition 5.1. A bifurcation occurs when a small, smooth change in a system parameter causes a sudden, topological change in the system's phase portrait, such as the creation, destruction, or stability change of equilibria or periodic orbits.

Example. A fundamental example of a bifurcation is known as a **saddle-node bifurcation**, a bifurcation which occurs when two fixed points of a dynamical system—one stable and one unstable—*collide and annihilate each other* as a parameter is varied, leading to a sudden appearance or disappearance of equilibrium behavior.
Consider the discrete-time dynamical system:

$$x_{n+1} = c + x_n^2,$$

where $c \in \mathbb{R}$ is a parameter.

Fixed Points. To find the fixed points, we set $x_{n+1} = x_n = x$:

$$x = c + x^2 \quad \Rightarrow \quad x^2 - x + c = 0.$$

This is a quadratic equation in x, and its discriminant is:

$$Discriminant = 1 - 4c.$$

- If $c < \frac{1}{4}$: There are two real fixed points (one stable, one unstable).
- If $c = \frac{1}{4}$: The two fixed points merge into one. This is the **saddle-node bifurcation** point.
- If $c > \frac{1}{4}$: There are no real fixed points.

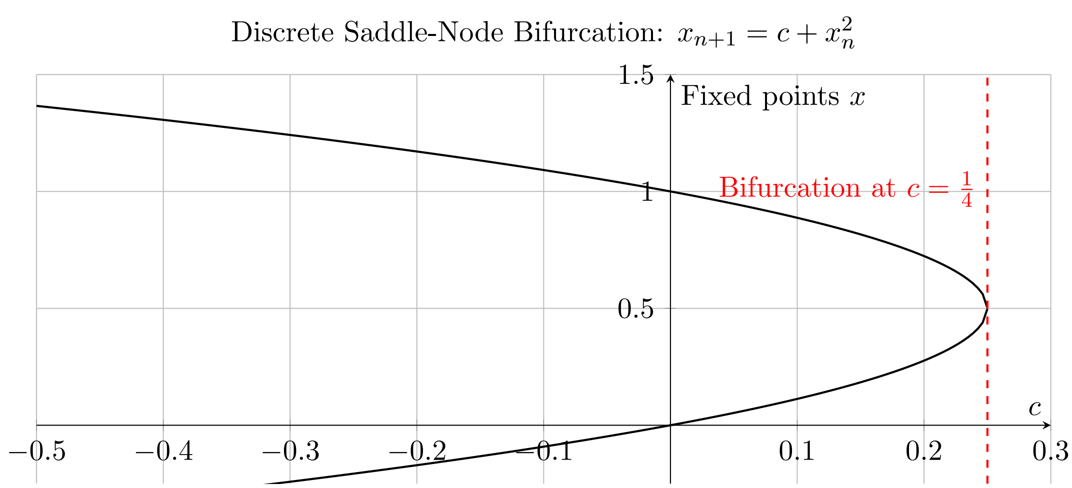


FIGURE 7. Bifurcation diagram of $x_{n+1} = c + x_n^2$ showing a saddle-node bifurcation at $c = \frac{1}{4}$

6. The Quadratic Family

We begin our investigation with quadratic functions because while iterations of linear functions are very predictable, iterations of quadratic functions are more complex.

Definition 6.1 (Quadratic Family)**.** Q_c is the family of quadratic functions such that $Q_c(x) = x^2+c$ and $c \in \mathbb{R}$.

We first seek to understand when fixed points exist in Q_c. Then, we wish to investigate whether these fixed points are attracting, repelling, or neutral. Afterwards, once we have an understanding of fixed points, or period 1 points, we can look at the behavior of period 2 cycles.

6.1. Fixed Points.

One way of understanding Q_c is to look at *the number of fixed points that exist* for different values of c.

In particular, we can check for solutions to the quadratic equation $Q_c = x$ or $x^2 + c = x$, the definition of a fixed point, *or* we can graph Q_c and then check for intersections with the line $y = x$. Using the quadratic formula, or $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, on $x^2 + c = x$ we can determine the value of c for which there are fixed points.

$$x^2 - x + c = 0$$
$$p_+ = \frac{1+\sqrt{1-4c}}{2}$$
$$p_- = \frac{1-\sqrt{1-4c}}{2}$$

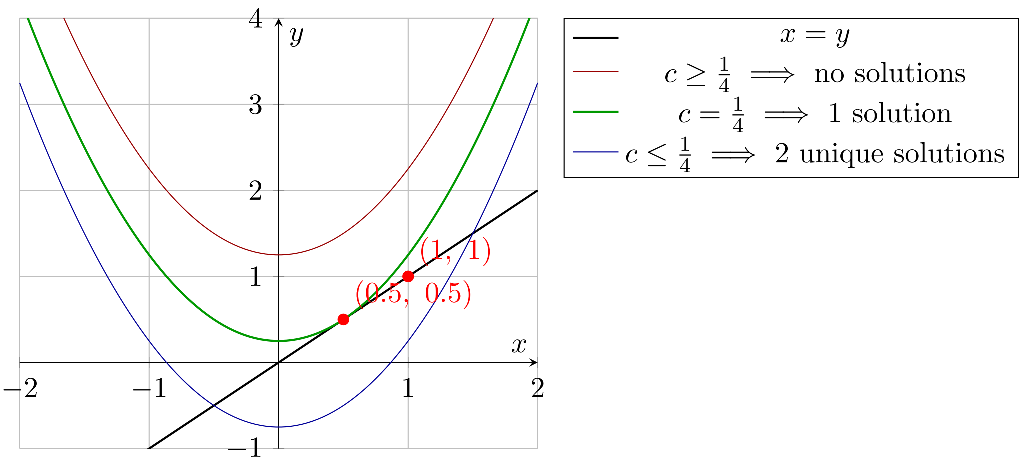
Definition 6.2. For clarity, p_+ will always refer to the fixed point on the right and p_- will always refer to the fixed point on the left.

We see that p_+ and p_- only exist if $1 - 4c \geq 0$. Thus, solving for c yields:

$$1 - 4c \geq 0 \implies 4c \leq 1 \implies c \leq 1/4$$

Therefore, we see that *fixed points* for $x^2 + c$ only exist when $c \leq \frac{1}{4}$. Now that we understand that we can begin to understand the behavior of fixed points when $c \leq \frac{1}{4}$. In particular, when $c = \frac{1}{4}$, there is one solution for Q_c. However, when $c < \frac{1}{4}$, there are two unique solutions.

By recalling that fixed points are just the intersections of a function with the line $y = x$, we can show this graphically.



Now that we have an understanding of when fixed points exist, we seek to understand what the behavior of those fixed points will be.

6.2. **Behavior of Fixed Points.** The behavior of fixed points for Q_c can be summarized as either attracting, repelling, or neutral. We can start by understanding what occurs at $c = \frac{1}{4}$ and then we can decrease our values of c and observe the behavioral changes to the fixed points.

6.2.1. $c = \frac{1}{4}$. We can simply compute the derivative of Q_c when $c = \frac{1}{4}$ to determine the behavior of the fixed point.

$$f(x) = x^2 + \frac{1}{4} \implies f'(x) = 2x$$
$$f'(\frac{1}{2}) = 1$$

The derivative at $x = \frac{1}{2}$ is 1, which means the single fixed point that exists is neutral. This is the most trivial case since only 1 fixed point exists, and the behavior of the fixed point is neither repelling or attracting.

6.2.2. $c < \frac{1}{4}$. Since $f'(x) = 2x$ and $p_+ = \frac{1+\sqrt{1-4c}}{2}$ and $p_- = \frac{1-\sqrt{1-4c}}{2}$:

$$f'(p_+) = 1 + \sqrt{1-4c}$$
$$f'(p_-) = 1 - \sqrt{1-4c}$$

When $c \leq \frac{1}{4}$ for p_+, the $\sqrt{1-4c}$ is always greater than 0 which means that $f'(p_+)$ is always ≥ 1. Thus, p_+ is repelling for $c \leq \frac{1}{4}$. However, for p_-, in order to find when the fixed point is attracting or repelling, we must solve $|f'(c)| < 1$:

$$\begin{aligned} -1 &< f'(p_-) < 1 \\ -1 < 1 &- \sqrt{1-4c} < 1 \\ 2 &> \sqrt{1-4c} > 0 \\ 3 &> -4c > -1 \\ -\frac{3}{4} &< c < \frac{1}{4} \end{aligned}$$

When $c \in (-\frac{3}{4}, \frac{1}{4})$, $|f'(c)| < 1$ which implies that p_- is attracting.

6.2.3. $c = -\frac{3}{4}$. Indeed we can check that when $c = -\frac{3}{4}$ that p_- is neutral.

$$\left|f'(p_-)\right| = \left|1 - \sqrt{1 - 4\left(-\frac{3}{4}\right)}\right| = \left|1 - \sqrt{4}\right| = |1-2| = 1 \implies p_- \text{ is neutral}$$

6.2.4. $c < -\frac{3}{4}$. We already understand that p_+ is repelling for $c \leq \frac{1}{4}$ so, we wish to understand what occurs for p_-. Using $f'(p_-) = 1 - \sqrt{1-4c}$, we see that when $c \leq -\frac{3}{4}$ the $\sqrt{1-4c}$ is always greater than 2 which means that $f'(p_-) < 1 - 2 \implies f'(p_-) < 1$. Thus, p_- is repelling for $c \leq -\frac{3}{4}$.

Our current understanding of Q_c can be summarized with this table:

$c > \frac{1}{4}$	no fixed points
$c = \frac{1}{4}$	one neutral fixed point
$-\frac{3}{4} < c < \frac{1}{4}$	p_+ is repelling; p_- is attracting
$c = -\frac{3}{4}$	p_+ is repelling; p_- is neutral
$c < -\frac{3}{4}$	two repelling fixed points

Definition 6.3. (Saddle-Node Bifurcation) A *bifurcation* refers to a division or split into two. A *saddle-node bifurcation* occurs at $c = \frac{1}{4}$ because for $c \geq \frac{1}{4}$, Q_c has no fixed points; for $c = \frac{1}{4}$, Q_c has one fixed point; but for $c \leq \frac{1}{4}$, the fixed points for Q_c splits into two.

Now that we not only have a sense of where fixed points exist, but the behavior of those fixed points when they exist, we seek to continue our understanding of Q_c by examining period 2 cycles.

6.3. **2-cycles.** We recall that fixed points for Q_c satisfy the function $x^2 + c = x$. Thus, points of period 2 cycles must satisfy $(x^2 + c)^2 + c = x$. After expanding we have:

$$x^4 + 2x^2c + c^2 + c = x \implies x^4 + 2x^2c - x + c^2 + c$$

Finding the solutions for a quartic polynomial is difficult however, we can realize that all fixed points are also solutions to this equation. For example a function defined by these 3 inputs:

$$f(x_0) = x_0$$
$$f(x_1) = x_2$$
$$f(x_2) = x_1$$

would have 3, 2-cycles namely:

$$f(f(x_0)) = f(x_0)$$
$$f(f(x_1)) = f(x_1)$$
$$f(f(x_2)) = f(x_2)$$

As such, since p_+ and p_- are fixed points they must also be solutions for our quartic polynomial. In other words $x^4 - 2xc - x + c^2 + c$ must be divisible by $(x - p_+)(x - p_-)$. However p_+ and p_- are just solutions of the equation $x^2 + c = x$ which means that $(x - p_+)(x - p_-) = x^2 - x + c$. As such, we can just find the quotient:

$$\frac{x^4 + 2x^2c - x + c^2 + c}{x^2 - x + c} = x^2 + x + c + 1$$

We can apply the same analysis of this function as with $x^2 - x + c$ to find the period 2 cycles exist. Applying the quadratic formula and simplifying we get:

$$q_{\pm} = \frac{1}{2}(-1 \pm \sqrt{-4c - 3})$$

In particular $q_{\pm}$ only exist in $\mathbb{R}$ if $-4c - 3 \geq 0$. Isolating c leaves:

$$-4c - 3 \geq 0 \implies -4c \geq 3 \implies 4c \leq -3 \implies c \leq -\frac{3}{4}$$

Definition 6.4 (Period Doubling Bifurcation)**.** We have a new kind of bifurcation occuring here. A *period doubling bifurcation* occurs at $c = -\frac{3}{4}$ because Q_c goes from only having cycles of period 1, to also having cycles of period 2.

We have discovered that period 2 cycles begin to exist when $c \leq -\frac{3}{4}$ and now, we hope to understand the behavior of these period 2 cycles.

6.4. **Behavior of Period 2 Cycles.** Just as fixed points can be repelling, attracting, or neutral. Periods of any cycle can also be repelling, attracting, or neutral.

We take the same approach as with a normal function. Namely, we can determine if cycle is attracting or repelling by taking the derivative of $x^4 + 2x^2c - x + c^2 + c$ or $f(f(x)) = x$. To find the attracting 2-cycles we can use the *chain rule* as a trick. The *chain rule* states that $f(f(x))' = f'(f(x)) \cdot f'(x)$ or the derivative of a composed function is the derivative of the outer function with respect to the inner function times the derivative of the outer function.

6.4.1. *Attracting Period 2 Cycles.*

$$\left|(Q_c \cdot Q_c)(q_+)'\right| < 1 \implies \left|Q_c'(Q_c(q_+))(Q_c'(q_+))\right| < 1$$

This looks intimidating, however we can apply our understanding of fixed points in order to substitute $Q_c(q_+) = q_-$ to simplify. We also have to recall that:

$$q_+ = \frac{1}{2}(-1 + \sqrt{-4c-3})$$
$$q_- = \frac{1}{2}(-1 - \sqrt{-4c-3})$$

$$Q_c'(q_-) = 2(q_-) = 2\left(\frac{1}{2}\left(-1 + \sqrt{-4c-3}\right)\right)$$
$$Q_c'(q_+) = 2(q_+) = 2\left(\frac{1}{2}\left(-1 - \sqrt{-4c-3}\right)\right)$$

$$\left|Q_c'(q_-)Q_c'(q_+)\right| < 1 \implies \left|4\left(\frac{1}{2}\left(-1 + \sqrt{-4c-3}\right)\right)\left(\frac{1}{2}\left(-1 - \sqrt{-4c-3}\right)\right)\right| < 1$$

We can apply our factorization trick of $(a-b)(a+b) = a^2 - b^2$ where $a = -1$ and $b = \sqrt{-4c-3}$.

$$\left|(-1)^2 - (\sqrt{-4c-3})^2\right| < 1 \implies |1 + 4c + 3| < 1 \implies |4c + 4| < 1 \implies |c + 1| < \frac{1}{4}$$

Next, we can resolve the absolute value and solve for c.

$$-\frac{1}{4} < c + 1 < \frac{1}{4} \implies -\frac{5}{4} < c < -\frac{3}{4}$$

As such, we can discover that a repelling 2 cycle exists for $x^2 - c$ when $-\frac{5}{4} < c < -\frac{3}{4}$.

6.4.2. *Repelling Period 2 Cycle.* We can apply a similar process to determine when $|(Q_c \cdot Q_c)(q_+)'| > 1$ because the process is identical up until $|4c + 4| > 1$, we will start there.

$$|c + 1| > \frac{1}{4} \implies c + 1 > \frac{1}{4} \text{ or } -c - 1 > \frac{1}{4} \implies c > -\frac{3}{4} \text{ or } c < -\frac{5}{4}$$

However, we can disregard $c > -\frac{3}{4}$ because 2-cycles do not even exist on this interval. Our current understanding of period 2-cycles in Q_c can be summarized with this table:

$c > -\frac{3}{4}$	no period 2 exists
$-\frac{5}{4} < c < -\frac{3}{4}$	attracting period 2 cycle
$c < -\frac{5}{4}$	repelling period 2 cycle

We have a very solid understanding of what occurs for Q_c in terms of fixed points, as well as period 2-cycles. Thus, we can continue to investigate what occurs when $c < -\frac{5}{4}$ by taking a look at *Chaos* and an example of a chaotic function, $x^2 - 2$.

6.5. **Chaos.**

Definition 6.5 (Chaos)**.** There are many ways to define chaos and no specific definition holds more merit than others. That being said, for the purposes of our exploration into the *The Quadratic Family* we will use two criteria to describe *chaos*.
A function F is *chaotic* if the following are true:

- F depends sensitively on initial conditions
- Periods of any value for F exist

As a simplified version of *chaos*, we can show that the function $x^2 - 2$ satisfies both conditions of a *chaotic* function.

6.5.1. F *depends sensitively on initial conditions.* As such, we can non-rigorously show that $x^2 - 2$ depends sensitively on initial conditions by simply, varying values of x and looking at the iterations.

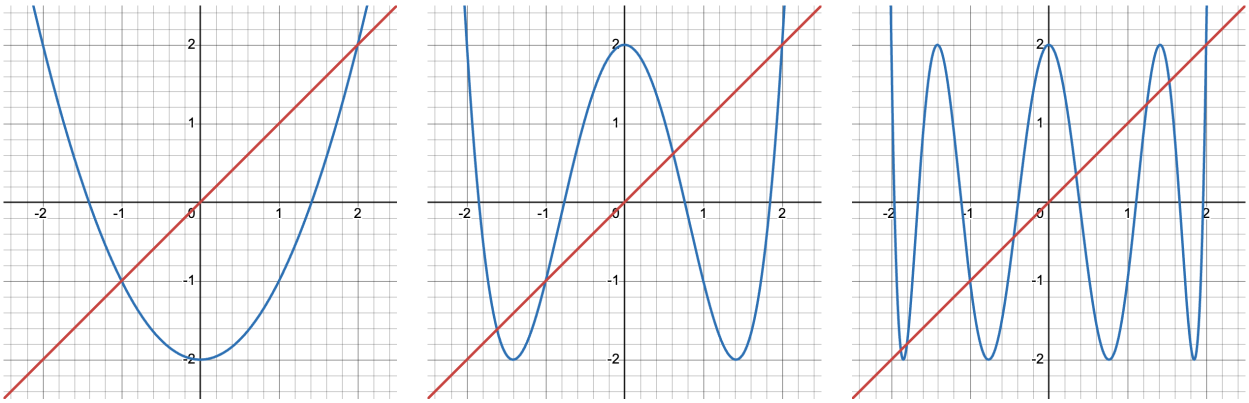
Iteration	$x_0 = 0.0$	$x_0 = 0.1$	$x_0 = 0.01$	$x_0 = 0.001$
1	0.000000	0.100000	0.010000	0.001000
2	-2.000000	-1.990000	-1.999900	-1.999999
3	2.000000	1.960100	1.999600	1.999996
4	2.000000	1.841992	1.998400	1.999984
5	2.000000	1.392935	1.993603	1.999936
⋮	⋮	⋮	⋮	⋮
95	2.000000	1.900693	1.003715	-0.220905
96	2.000000	1.612633	-0.992556	-1.951201
97	2.000000	0.600585	-1.014832	1.807186
98	2.000000	-1.639297	-0.970116	1.265920
99	2.000000	0.687296	-1.058876	-0.397447
100	2.000000	-1.527624	-0.878782	-1.842036

We can observe that even small changes to the initial conditions makes it impossible to predict its orbit. In particular, predicting the behavior of a chaotic function by computation will eventually be futile. This is because the error accumulated due to rounding and truncation will eventually compound and since the function depends sensitively on its initial conditions, these small discrepancies make it impossible to predict the output of the function with complete accuracy.

6.5.2. *Periods of any value for* F *exist.* We can take a similarly non-rigorous approach and show the existence of many different periods graphically. For clarity, we will refer to $x^2 - 2$ as $f(x)$.

The intersections between the graph of a function and the line $y = x$ is the points at which the function is equal to x.

$$\text{Intersections between } f(x) \text{ and } y = x \implies \text{fixed points}$$
$$\text{Intersections between } f(f(x)) \text{ and } y = x \implies \text{period 2 points}$$
$$\text{Intersections between } f(f(f(x))) \text{ and } y = x \implies \text{period 3 points}$$



(A) $f(x)$ (B) $f(f(x))$ (C) $f(f(f(x)))$

It is clear that there is something strange going on with the periodic points for $x^2 - 2$. In particular Q_c goes from just having period 2 cycles at $c = -\frac{3}{4}$ to cycles of any arbitrary length at $c = -2$. It appears as though new periodic points are being created after each iteration of the function. Each valley or dip of the function $x^2 - 2$ down to -2 is being duplicated for each successive iteration. As such, this results in 2^n periodic points of period n. In other words, there are periodic points for any arbitrary period.

Our investigation of the Quadratic Family has discussed the existence and behavior of both fixed points, and period 2 cycles. In addition, we have looked at the behavior of the chaotic function $x^2 - 2$. However, our understanding so far has been limited by values of c that are real. The next step in our investigation is to extend c into the complex numbers.

7. The Julia Set and the Mandelbrot Set

Definition 7.1. A complex number is a number $z = a + bi$ where $a, b \in \mathbb{R}$ and i is defined by $i^2 = -1$.

Definition 7.2. The complex conjugate of a complex number $z = a + bi$ is $\overline{z} = a - bi$.

Some properties of complex numbers:

$$\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
(a+bi)(c+di) &= (ac-bd)+(ad+bc)i \\
|z| = |a+bi| &= \sqrt{a^2+b^2} \\
z\overline{z} &= |z|^2 \\
e^{i\theta} &= \cos\theta + i\sin\theta \\
z &= r(\cos\theta + i\sin\theta) = re^{i\theta} \\
z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
z^{1/n} &= r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \ldots, n-1
\end{aligned}$$

Note that the last property implies that every nonzero complex number has n *distinct* nth roots.

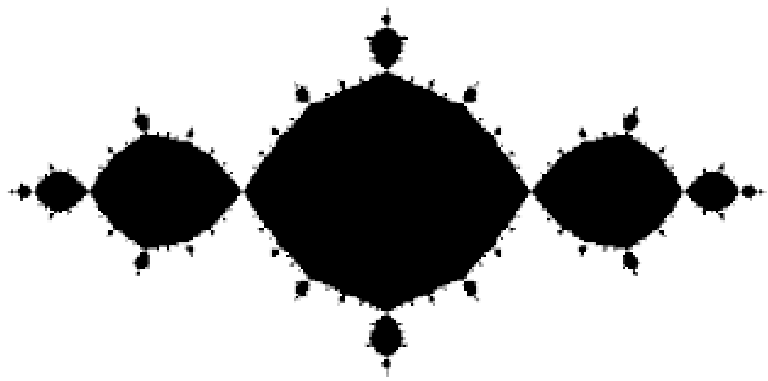
We have already discussed the function $f(x) = x^2 + c$ defined on $\mathbb{R}$. Now we can consider $f(z) = z^2 + c$ defined over the complex plane.

Definition 7.3. Consider some complex number w and some complex function $f(z) = z^2 + c$. We say that the orbit of w on f remains bounded if there exists some M such that $|f^n(w)| < M$ for all n. Here, $|z|$ is defined as $|z| = |a + bi| = |\sqrt{a^2 + b^2}$.

Now for any complex function, we can determine which complex numbers remain bounded when iterated with the function, and which don't. One interesting object we can consider is the set of complex numbers that remain bounded when iterated.

Definition 7.4. The filled Julia Set for some $f(z) = z^2 + c$ is defined as the set of all complex numbers that remain bounded when iterated by f. It is denoted by K_c.

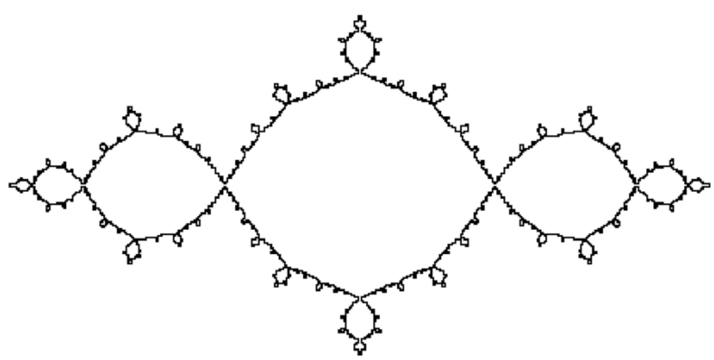
We can visualize the filled Julia set by coloring all complex points in the set.
Here is an image of the filled Julia Set for $f(z) = z^2 - 1$:
https://www.researchgate.net/figure/The-filled-Julia-set-of-f-z-z-2-1_fig1_311926148



Definition 7.5. The Julia Set is the boundary of the filled Julia set. It is denoted by J_c.

Here is the Julia set for $f(z) = z^2 - 1$:

https://www.researchgate.net/figure/The-Julia-set-of-the-polynomial-z-2-1_fig3_2110181



Some properties of K_c:

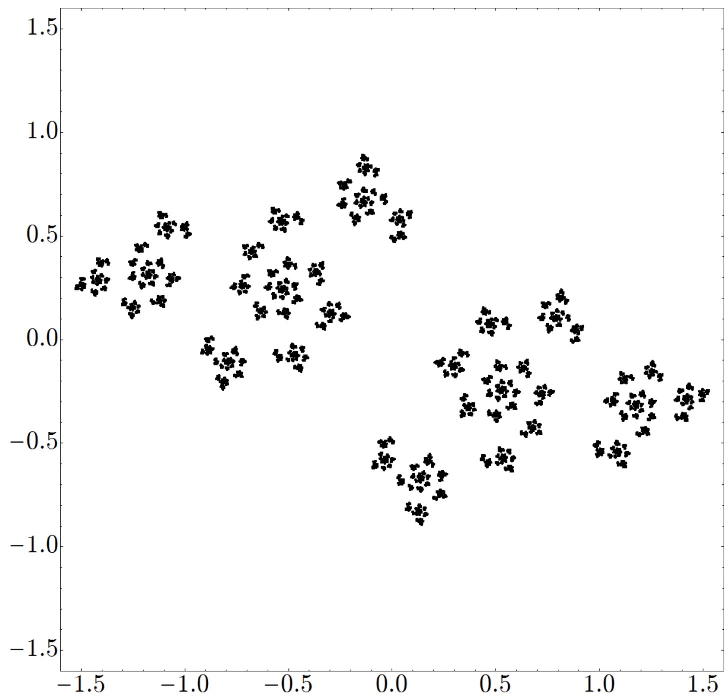
1. K_c is connected if the orbit of 0 is bounded and it is totally disconnected if not.
2. K_c is quasi self-similar, meaning slightly modified copies of the set are found in other parts of the set, just scaled and shifted.

The first of those properties is a theorem known as The Fundamental Dichotomy.

We say that K_c is connected when the set is all in one piece and contains no gaps or separations. Formally, this means that K_c cannot be split into two or more non-empty, disjoint open subsets.

We say that K_c is totally disconnected if it consists only of isolated points and contains no connected subset that contains more than a single point.

Above was a visualization of a connected K_c.
Here is a totally disconnected K_c (The filled Julia Set for $f(z) = z^2 - 0.75 + 0.25i$):
(https://e.math.cornell.edu/people/belk/dynamicalsystems/NotesJuliaMandelbrot.pdf)

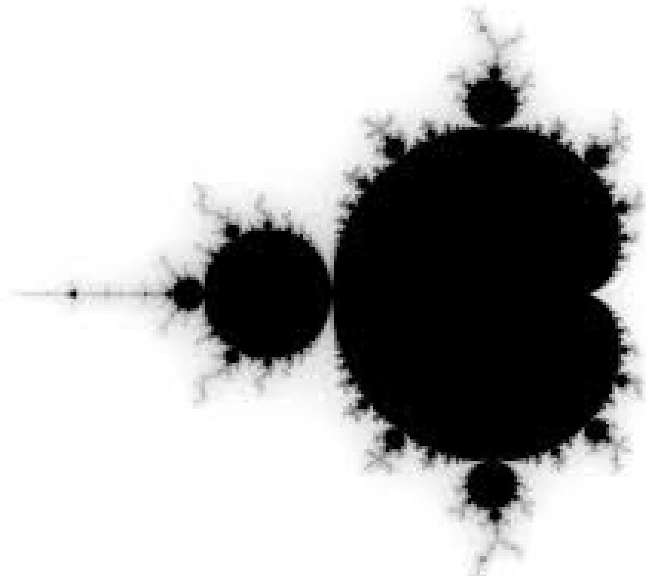


Fact. *The orbit of any initial value with* $f(z) = z^2 + c$ *diverges to infinity if* $|c| > 2$.

Definition 7.6. The Mandelbrot set M is the set containing all values of c for which K_c is connected.

To compute the Mandelbrot Set, we can choose some upper bound N on the number of iterations we will check. For every complex c we can iterate $f(z) = z^2 + c$, starting with 0. If the magnitude of the value ever becomes greater than 2, we conclude c is not in M. If we have performed N iterations and the orbit of 0 remained bounded, we can say $c \in M$.

We can visualize the Mandelbrot Set the same way we did for the Julia Set:
`https://paulbourke.net/fractals/mandelbrot/`



The largest region in the image is referred to as the *main cardioid*. This region contains the only values of c that give $z^2 + c$ an attracting fixed point. The boundary of the region is the values of c that give $z^2 + c$ a neutral fixed point.

To find the region of the *main cardioid* algebraically, we must find when $f(z) = z^2 + c$ has an attractive fixed point. We first must solve the quadratic $z^2 + c = z$ which yields the fixed points $z = \frac{1 \pm \sqrt{1-4c}}{2}$. A fixed point is attractive when $|f'(z)| < 1$ so we must find when $|1 \pm \sqrt{1-4c}| < 1$. The solution to this inequality is messy but yields the desired region.

The next region to note is the *period 2 bulb*, which is the second largest region, just to the left of the main cardioid. As the name implies, this is the region that contains all values of c where f has an attracting period 2 cycle.

This region can be computed by solving $f(f(z)) = z$ and only considering solutions that aren't also fixed points. Doing this yields the equation for a circle, which is in fact, the *period 2 bulb*:

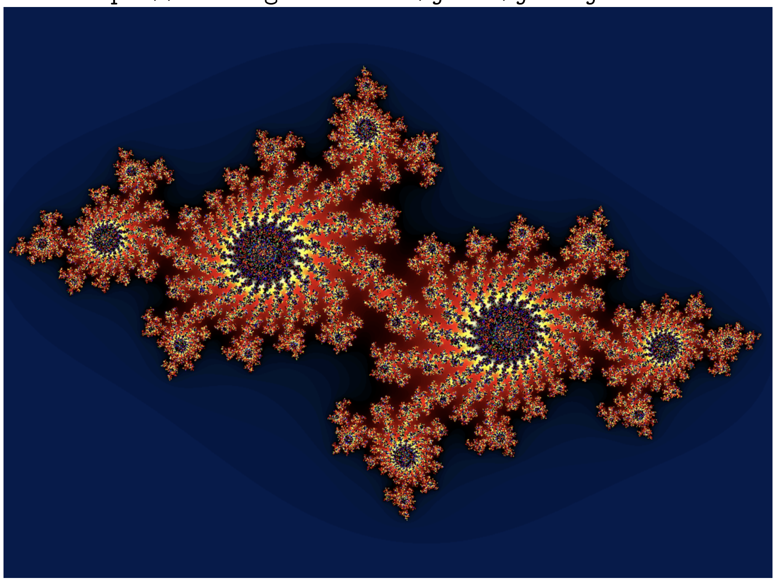
$$\begin{aligned} f(f(z)) &= z \\ (z^2 + c)^2 + c &= z \\ z^4 + 2cz^2 + c^2 + c &= z \\ z^4 + 2cz^2 - z + (c^2 + c) &= 0 \end{aligned}$$

This can be factored to yield $(z^2 - z + c)(z^2 + z + c + 1) = 0$. Since we already know that $(z^2 - z + c)$ represents our fixed points, we can solve the other terms to get our points of period 2. Using the quadratic formula on $z^2 + z + c + 1$ we get the period 2 points of $z = \frac{-1 \pm \sqrt{-4c-3}}{2}$.

To find when these fixed points are attracting, we must find when $|(f \circ f)'(z_1)| < 1$ where z_1 is one of the two period 2 points we computed. Using the same method as previously used we can simplify our inequality to $|c+1| < \frac{1}{4}$ which is a circle centered at $-1+0i$ with a radius of $\frac{1}{4}$, which is what defines the period 2 bulb.

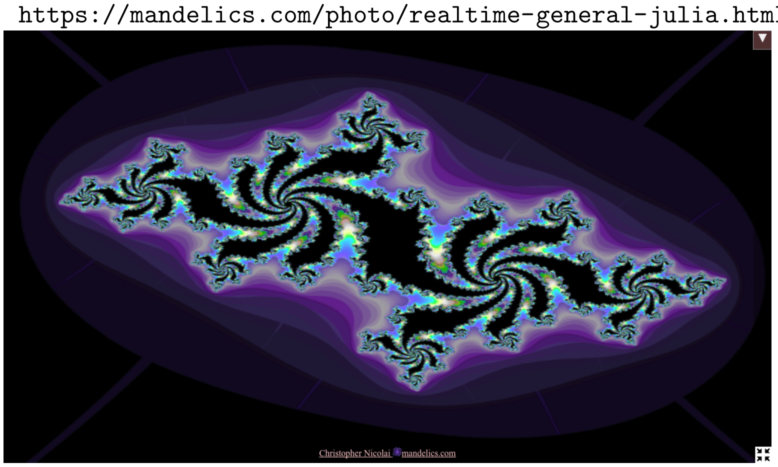
Below are images of several more interesting Julia sets:

`https://www.mcgoodwin.net/julia/juliajewels.html`



c=0.687 + 0.312i

`https://mandelics.com/photo/realtime-general-julia.html`



c=-0.6078 + 0.438i

Acknowledgements

Sections 1-5 were written by Marko Mano, Section 6 by Derek Yin, and 7 by Kalle Nikula-Gill. The authors acknowledge Sean Li for his valuable mentorship throughout the program. Additionally, the authors are truly grateful for the guidance and support from Mary Stelow, Marisa Gaetz, and the MIT PRIMES Circle Program.