Connectedness and Cycle Spaces of Friends-and-Strangers Graphs

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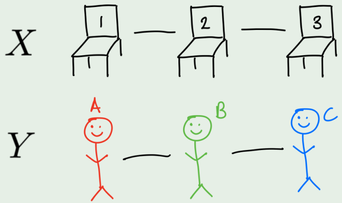
The Friends-and-Strangers Graph

Example (Friends and Chairs Graphs)

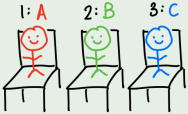
- Consider graphs X, Y with n vertices each.
- Treat the vertices of X as "chairs" and the vertices of Y as "people."
- Two chairs are adjacent in X if and only if they are "next to" each other.
- Two people are adjacent in Y if and only if they are friends with each other.



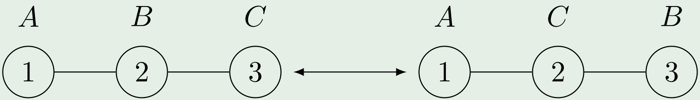
The Friends-and-Strangers Graph

Example (Swaps)

- Consider an arrangement of the n friends sitting in the chairs (or bijections mapping chairs in $V(X)$ to people in $V(Y)$).



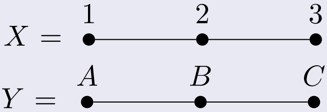
- For any arrangement, two people can swap seats if they are friends (adjacent in Y) and sitting in chairs next to each other (adjacent in X).
- Such a swap is called an *(X, Y)-friendly swap*.
- Can always swap back!



The Friends-and-Strangers Graph

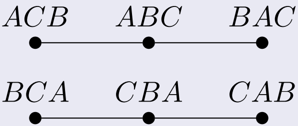
Definition (Defant–Kravitz, 2020)

- The friends-and-strangers graph $\mathsf{FS}(X, Y)$ contains all $n!$ of these seating arrangements as its vertices.
- Two vertices in $\mathsf{FS}(X, Y)$ are adjacent if and only if there is a (X, Y)-friendly swap between them.



- We have the following graph for $\mathsf{FS}(X, Y)$:

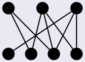
$\mathsf{FS}(X, Y) =$



Definitions

Definition

A graph G is *bipartite* if we can split $V(G)$ into two sets such that no edges exist between vertices in the same set.



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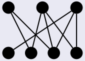
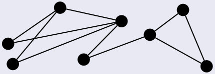
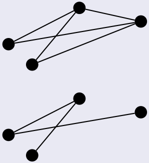


Definition

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph is *induced* if every edge in G between vertices in H is also in H.





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A graph G is *connected* if for all $a, b \in V(G)$, there is a path of edges in G leading from a to b. A *connected component* of G is a maximal connected subgraph of G.

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Definition

A graph is *biconnected* if every induced subgraph with one vertex removed is connected.

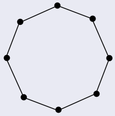
Example

Special Graphs

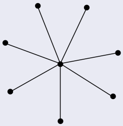
Definition

- The **path graph** Path_n:

- The **cycle graph** Cycle_n:



- The **star graph** Star_n:

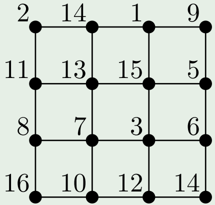


- The graph Path_n has n vertices, etc.

Application

Example (15-puzzle)

2	14	1	9
11	13	15	5
8	7	3	6
	10	12	14

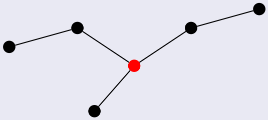


- Moves in the 15-puzzle are equivalent to $\mathsf{FS}(\mathsf{Star}_{16}, \mathsf{Grid}_{4\times 4})$-friendly swaps.
- The graphs Star_{16} and $\mathsf{Grid}_{4\times 4}$ are both bipartite, which means $\mathsf{FS}(\mathsf{Star}_{16}, \mathsf{Grid}_{4\times 4})$ is disconnected.
- Therefore, the 15-puzzle is not always solvable.

Spiders and Dandelions

Definition

A *spider* is a disjoint union of paths connected to a single *center* vertex c. The paths that result from deleting c are called the *legs* of the spider. The number of vertices in a leg is called its *length*. We write $\mathsf{Spider}(\lambda_1, \ldots, \lambda_k)$ for the spider with legs of lengths $\lambda_1, \ldots, \lambda_k$.



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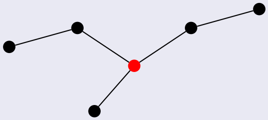
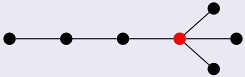


Definition

The *dandelion* graph $\mathsf{Dand}_{k,n}$ is a spider with $k-1$ legs of length 1 and 1 leg of length $n-k$.



Dandelions

Question

When is the graph $\mathsf{FS}(\mathsf{Dand}_{k,n}, Y)$ *connected?*

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Example

Note that $\mathsf{Dand}_{2,n} = \mathsf{Path}_n$. It is known that $\mathsf{FS}(\mathsf{Path}_n, Y)$ is only connected if Y is the complete graph on n vertices, K_n.

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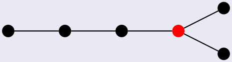
Theorem (Defant–Kravitz, 2020)

For all $n \geq 5$, $\mathsf{FS}(\mathsf{Dand}_{3,n}, Y)$ *is connected if and only if every vertex of* Y *has degree at least* $n - 2$.



Dandelions

Theorem (DDLW, 2022)

If $n \geq 2k - 1$, then $\mathsf{FS}(\mathsf{Dand}_{k,n}, Y)$ is connected if and only if every induced subgraph of Y with k vertices is connected.

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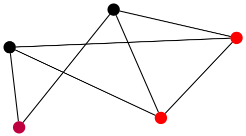
Sanity Check:

- If $k = 2$, then every induced subgraph with 2 vertices is connected, implying that every vertex neighbors every other
- If $k = 3$, then every induced subgraph with 3 vertices is connected, so no vertex can have ≥ 2 neighbors that it is not connected with.



Dandelions

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What if $n < 2k - 1$?

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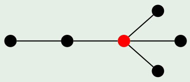
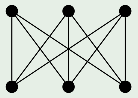
Generally Unsolved!

Example ($n < 2k - 1$)

Let $X = \mathsf{Dand}(4, 6)$ and Y be $K_{3,3}$, as shown on the right. Every induced subgraph of 4 vertices in Y is connected, but X and Y are both bipartite, so $\mathsf{FS}(X, Y)$ is disconnected.





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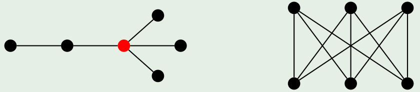

Theorem (Wilson, 1974)

Note that $\mathsf{Star}_n = \mathsf{Dand}_{n-1,n}$. *Then,* $\mathsf{FS}(\mathsf{Star}_n, Y)$ *is connected if* Y *is biconnected, not bipartite, and not* Cycle_n.

Fruit Graphs and Hereditary Families

Definition

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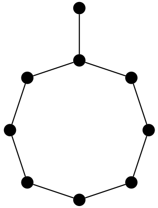
A family $\mathcal{Y}$ of (isomorphism classes of) graphs is called *hereditary* if it is closed under taking induced subgraphs (i.e., every induced subgraph of a graph in $\mathcal{Y}$ is also in $\mathcal{Y}$).



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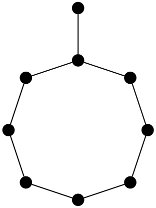
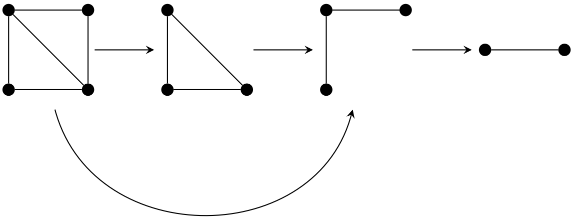
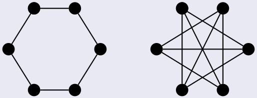




Complements and An Important Lemma

Definition

The *complement* $\overline{G}$ of a graph G is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G})$ consists of all edges between two vertices in $V(G)$ not in $E(G)$.



Lemma

Let $\mathcal{Y}$ be a hereditary family of connected graphs. Let X be a graph with n vertices such that $\mathsf{FS}(X, Y)$ is connected for every n-vertex graph $Y \in \mathcal{Y}$. Let $x \in V(X)$, and let X' be the graph obtained from X by adding a new vertex x' together with the edge $\{x, x'\}$. Then $\mathsf{FS}(X', Y')$ is connected for every $(n+1)$-vertex graph $Y' \in \mathcal{Y}$.

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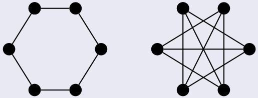

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Corollary of Important Lemma

The only n-vertex induced subgraph of $\overline{\mathsf{Cycle}_{n+1}}$ is $\overline{\mathsf{Path}_n}$.

Furthermore $\overline{\mathsf{Cycle}_n}$ is a subgraph of $\overline{\mathsf{Path}_n}$.

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Corollary

Let X have $n \geq 5$ vertices such that $\mathsf{FS}(X, \overline{\mathsf{Cycle}_n})$ is connected. Then $\mathsf{FS}(X', \overline{\mathsf{Cycle}_{n+1}})$ is connected.

Spiders and Complements of Cycles or Fruits

Theorem (DDLW, 2022)

Let $n \geq 4$. *The graph* $\mathsf{FS}(\mathsf{Spider}(\lambda_1, \ldots, \lambda_k), \overline{\mathsf{Cycle}_n})$ *is connected if and only if* $(\lambda_1, \ldots, \lambda_k)$ *is not of the form* $(\lambda_1, 1, 1)$ *and is not in the following list:*

$$(1,1,1,1), \ (2,2,1), \ (2,2,2), \ (3,2,1), \ (3,3,1), \ (4,2,1), \ (5,2,1).$$

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In general, if the legs are long enough or there are enough legs, we get a connected friends-and-strangers graph.

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In general, if the legs are long enough or there are enough legs, we get a connected friends-and-strangers graph.

Theorem (DDLW, 2022)

The graph $\mathsf{FS}(\mathsf{Spider}(\lambda_1, \ldots, \lambda_k), \overline{\mathsf{Cycle}_n^{\perp}})$ for $k \geq 3$ is disconnected if and only if $(\lambda_1, \ldots, \lambda_k)$ is of one of the following forms:

$$(\lambda_1, 1, 1, 1), \quad (\lambda_1, \lambda_2, 1), \quad (2,2,2).$$

What does a friends-and-strangers graph actually look like? Cycles can help us understand the structure.

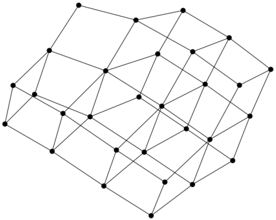
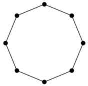


Figure 1: One connected component of $\mathsf{FS}(\mathsf{Cycle}_8, \mathsf{Dand}_{3,8})$.

$\mathsf{Cycle}_8 =$ 

$\mathsf{Dand}_{3,8} =$ 

Cycle Spaces

Definition

A graph is called **even-degree** if each of its vertices has even degree. An **edge-subgraph** of a graph G is a subgraph of G that has the same vertex set as G. Given edge-subgraphs H and H' of G, let $H \triangle H'$ be the edge-subgraph of G whose edge set is the symmetric difference of the edge sets of H and H'. The **cycle space** of G is the set of all even-degree edge-subgraphs; it is a vector space over the 2-element field $\mathbb{F}_2$ in which the addition operation is given by the symmetric difference $\triangle$.

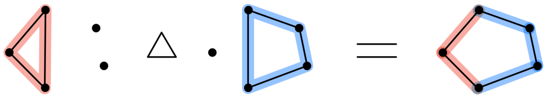


Figure 2: Symmetric difference operation.

It is well known that the cycle space of a graph is spanned by its cycles.

Cycle Space of $\mathsf{FS}(\mathsf{Path}_n, Y)$

Michael Naatz proves the following result about the cycle space of $\mathsf{FS}(\mathsf{Path}_n, Y)$, where Y is **any** n-vertex graph.

Theorem (Naatz, 2000)

If Y is any n-vertex graph, then the cycle space of $\mathsf{FS}(\mathsf{Path}_n, Y)$ is spanned by 4-cycles and 6-cycles.

Can we find an analogous result about the cycle space of $\mathsf{FS}(\mathsf{Cycle}_n, Y)$?

Isometric Cycles

We can take a similar approach as Naatz by studying isometric cycles.

Definition

An **isometric cycle** of a graph G is a subgraph H of G that is a cycle and has the property that for all vertices u and v of H, the distance between u and v in H is the same as the distance between u and v in G.

- Every non-isometric cycle can be written as a symmetric difference of smaller isometric cycles.
- Each isometric cycle is a symmetric difference of 4 and 6-cycles.
- Key idea: Each label in the label sequence of a shortest path in $\mathsf{FS}(\mathsf{Path}_n, Y)$ is unique (no friendly swaps are repeated).

$\mathsf{FS}(\mathsf{Path}_n, Y)$ to $\mathsf{FS}(\mathsf{Cycle}_n, Y)$

How can we extend results for $\mathsf{FS}(\mathsf{Path}_n, Y)$ to $\mathsf{FS}(\mathsf{Cycle}_n, Y)$? We need certain restrictions on Y.

Definition

A **dominating set** of a graph G is a subset D of the vertex set of G such that every vertex in G is either in D or is adjacent to a vertex in D. The minimum size of a dominating set of G is called the **domination number** of G.

When the domination number of Y is at least 3 (no 2 elements are adjacent to every other element in Y), it turns out that $\mathsf{FS}(\mathsf{Cycle}_n, Y)$ behaves very similarly as $\mathsf{FS}(\mathsf{Path}_n, Y)$ because we can show that no shortest path has any repeated friendly swaps.

Theorem for $\mathsf{FS}(\mathsf{Cycle}_n, Y)$

Theorem (DDLW, 2022)

Let Y be an n-vertex graph with domination number at least 3. The cycle space of $\mathsf{FS}(\mathsf{Cycle}_n, Y)$ is spanned by 4-cycles and 6-cycles. If Y is triangle-free, then the cycle space of $\mathsf{FS}(\mathsf{Cycle}_n, Y)$ is spanned by 4-cycles.

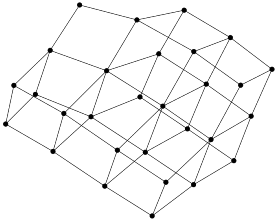


Figure 3: One connected component of $\mathsf{FS}(\mathsf{Cycle}_8, \mathsf{Dand}_{3,8})$.

What if Y has domination number 1 or 2?

For domination number 2, the theorem initially appears to hold, but it is not true in general.

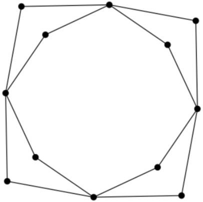


Figure 4: O

ne connected component of $\mathsf{FS}(\mathsf{Cycle}_4, \mathsf{Cycle}_4)$.

When Y has domination number 1, the theorem does not hold. Example: when $Y = \mathsf{Star}_n$, $\mathsf{FS}(\mathsf{Cycle}_n, \mathsf{Star}_n)$ consists of $(n-2)!$ connected components isomorphic to $\mathsf{Cycle}_{n(n-1)}$.)

Acknowledgements

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- The PRIMES Organizers
- Our parents
- Everyone for listening!

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