Different Fields in Combinatorial Optimization

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Combinatorial optimization is the study of finding optimal solutions (e.g. minimum cost, maximum profit) to a diverse set of mathematical problems.

Examples and applications include:

- logistics + scheduling,
- artificial intelligence,
- the traveling salesman problem,
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Linear Programming

Canonical Form of LPs

Suppose we have a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Canonical form of Linear Programs

Find a vector $x \in \mathbb{R}^n$ such that:

$$\begin{aligned} \text{Maximize} \quad & Z = c^T x. \\ \text{Subject to} \quad & Ax \leq b \\ & x \geq 0. \end{aligned}$$

- A feasible vector is any $x \geq 0$ for which $Ax \leq b$ holds.
- The LP is unbounded if the maximum value is not finite.
- An optimum vector is any feasible vector that gives a maximum value.

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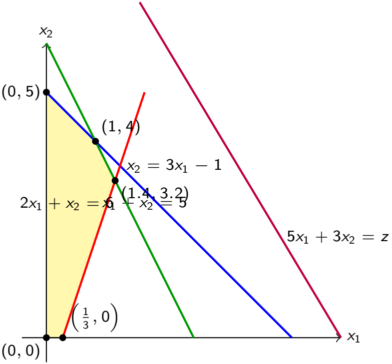
Duality: An Example

Example

Consider the following LP (The Primal Problem):

$$\begin{aligned} \text{Maximize} \quad & z = 5x_1 + 3x_2. \\ \text{Subject to} \quad & x_1 + x_2 \le 5 \\ & 3x_1 - x_2 \le 1 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0. \end{aligned}$$



From the graph, we can see that the maximum value of z is achieved at $(1, 4)$.

Duality: An Example (cont.)

Example

- To deduce an upper bound, we combine the constraints using nonnegative multipliers y_i:

$$\left.\begin{aligned} y_1(x_1 + x_2) &\leq 5y_1 \\ y_2(3x_1 - x_2) &\leq 1y_2 \\ y_3(2x_1 + x_2) &\leq 6y_3 \end{aligned}\right\} \quad \text{for all } y_1, y_2, y_3 \geq 0.$$

The sum then becomes:

$$y_1(x_1 + x_2) + y_2(3x_1 - x_2) + y_3(2x_1 + x_2) \leq 5y_1 + 1y_2 + 6y_3.$$

Duality: An Example (cont.)

Example

- This form is equivalent to:

$$(y_1 + 3y_2 + 2y_3) \cdot x_1 + (y_1 - y_2 + y_3) \cdot x_2 \leq 5y_1 + y_2 + 6y_3.$$

- Therefore, $5y_1 + y_2 + 6y_3$ is an upper bound of said LP, as long as:

$$\begin{cases} y_1 + 3y_2 + 2y_3 \geq 5 \\ y_1 - y_2 + y_3 \geq 3. \end{cases}$$

- Finding the best upper bound is equivalent to minimizing:

$$5y_1 + y_2 + 6y_3.$$

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Duality: An Example (cont.)

Example

- Thus we get a new LP:

$$\begin{aligned} \text{Minimize} \quad & z = 5y_1 + y_2 + 6y_3. \\ \text{Subject to} \quad & y_1 + 3y_2 + 2y_3 \geq 5 \\ & y_1 - y_2 + y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

- This motivates the definition of a dual LP.

Dual Linear Programs

Definition

Given initial LP $\max\{c^T x : Ax \leq b, x \geq 0\}$, we define the dual LP as

$$\begin{aligned} \text{Minimize} \quad & y^T b. \\ \text{Subject to} \quad & y^T A \geq c^T \\ & y \geq 0. \end{aligned}$$

The initial LP is then called the primal LP.

- The roles for the primal LP and its dual are symmetric:

Properties

The dual of the dual of an LP is equivalent to the original LP.

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Weak Duality Theorem

We note that solutions of dual LPs yield upper bounds for primal LPs. This is known as weak duality:

Theorem (Weak Duality)

For any primal feasible solution x and any dual feasible solution y,

$$c^T x \leq y^T b.$$

Hence, if $c^T x = y^T b$ then x and y are optimal. If either is unbounded, the other is therefore infeasible.

Strong Duality Theorem

Question

Can solutions of dual LPs yield optimum values of primal LPs?

The strong duality theorem confirms our answer.

Theorem (Strong Duality)

The primal LP has an optimum solution x^* if and only if the dual LP has optimum solution y^*. In this case, they have the same target value $c^T x^* = b^T y^*$.

Complementary Slackness Conditions

- Let x and y be the optimum solution to the primal and dual LPs, by the strong duality theorem:

$$c^T x = y^T b.$$

- Note that the following must hold:

$$\begin{cases} y^T(Ax - b) = 0 \\ (y^T A - c^T)x = 0. \end{cases}$$

  Those conditions are called complementary slackness conditions.

- Conversely, if some primal feasible x and dual feasible y satisfies the above, then they are optimum.

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Complementary Slackness Conditions: An Example

Example

For our previous examples, the complementary slackness conditions read:

$$\begin{cases} y_1(x_1 + x_2 - 5) = 0 \\ y_2(3x_1 - x_2 - 1) = 0 \\ y_3(2x_1 + x_2 - 6) = 0, \end{cases} \qquad \begin{cases} x_1(y_1 + 3y_2 + 2y_3 - 5) = 0 \\ x_2(y_1 - y_2 + y_3 - 3) = 0. \end{cases}$$

- We note that $(x_1, x_2) = (1, 4)$ and $(y_1, y_2, y_3) = (1, 0, 2)$ satisfy the above equation. Therefore the optimum value must be 17.
- Also note that $y_2 = 0$, which also implies that the second constraint in the primal LP

$$3x_1 - x_2 \le 1,$$

is not used when arriving at the optimum.

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Applications: The Transport Problem

Let I be a set of supply warehouses and J be a set of demand warehouses.

- λ_i: initial supply of origin $i \in I$.
- μ_j: required demand of destination $j \in J$.
- c_{ij}: the cost of transporting one unit of goods from i to j.

By making a plan to transport $x_{ij} \geq 0$ goods from i to j, we want to minimize the total cost:

$$C = \sum_{i \in I,\ j \in J} c_{ij} \cdot x_{ij},$$

such that the supply matches the demand:

$$\sum_{k \in J} x_{ik} = \lambda_i, \quad \sum_{\ell \in I} x_{\ell j} = \mu_j.$$

This is called the Optimal Transport Problem.

Applications: The Transport Problem

- The dual LP can be then formulated as:

$$\begin{aligned} \text{Maxmize} \quad & \sum_{i\in I} \lambda_i \cdot y_i + \sum_{j\in J} \mu_j \cdot z_i. \\ \text{Subject to} \quad & y_i + z_j \le c_{ij} \qquad \forall i \in I,\ j \in J. \end{aligned}$$

  By the duality theory, solving the optimal transport problem is equivalent to solving this dual problem.

- When $I = J$ is a set of vertices of a graph and c_{ij} is the distance between these two nodes, they satisfy triangle inequalities

$$c_{ij} + c_{jk} \ge c_{ik}.$$

  In this case, the dual LP can be simplified to a LP of $|I|$ variables.

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Algorithms for Solving LPs

Algorithm	Strategy	Pros/Usage	Cost per iteration	Complexity
Simplex	Moves along vertices	Small/Medium LPs	simple to implement	$O(2^n)$
Interior Point	Paths through interior	Large/Sparse LPs	computationally more expensive	$O(n^3 \cdot L)$
Ellipsoid	Shrinks ellipsoids	Theoretic Proofs	very expensive	$O(n^6 \cdot L)$

Matroid Theory

Independence Systems and Matroids

Definition

Let M be a independence system with ground set $E(M)$ and the set of independent sets $\mathcal{I}(M) \subseteq \mathcal{P}(E(M))$. If

1. $\emptyset \in \mathcal{I}(M)$,
2. and for all $X \subset Y$ and $Y \in \mathcal{I}(M)$, then $X \subset \mathcal{I}(M)$.

The second condition is known as the hereditary axiom.

Definition

Let M be a matroid with ground set E and the set of independent sets $\mathcal{I}(M) \subseteq \mathcal{P}(E(M))$. Then, M is an independence system that also satisfies

- For all $X, Y \in \mathcal{I}(M)$ with $|Y| > |X|$, there exists $e \in Y \backslash X$ such that $X + e \in \mathcal{I}(M)$.

This property is known as the exchange axiom.

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Examples

Vector space

Let V be a vector space over a field $\mathbb{F}$. Let $E \subseteq V$ be a finite set of vectors. Define $\mathcal{I}$ as the collection of all sets of linearly independent vectors in E. Then $M = (E, \mathcal{I})$ forms a matroid.

Sketch of proof

The first two conditions are immediate. To see exchange axiom, suppose that $X + e \notin \mathcal{I}(M)$ for every $e \in Y \setminus X$. This implies that $\mathrm{Im}(Y) \subseteq \mathrm{Im}(X)$ so $|Y| \leq |X|$. Contradiction.

Example

Let $E = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^2$ where

$$v_1 = \langle 1, 0\rangle, v_2 = \langle 0, 1\rangle, v_3 = \langle 1, 1\rangle, v_4 = \langle 2, 0\rangle.$$

Thus,

$$\mathcal{I} = \{\{\emptyset\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}.$$

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Examples

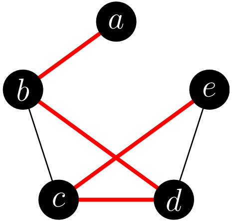
Graphic Matroid

Let $G = (V, E)$ be a graph. Let

$$\mathcal{I} = \{F \subseteq E \colon F \text{ is acyclic}\},$$

i.e. $\mathcal{I}$ is the set of forests. Define the graphic matroid by $M = (E, \mathcal{I})$.

Example



We have

$$E = \{ab, bc, bd, cd, ce, de\}$$

and

$$\begin{aligned} \mathcal{I} = \{&\{ab, bc, cd, de\}, \{ab, bc, ce, de\}, \\ &\{ab, bd, cd, ce\}, \{ab, bd, de, ce\}, \\ &\ldots\}. \end{aligned}$$

Rank

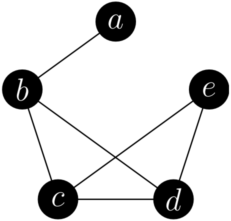
Definition

Consider any independence system M. Then, there exists a rank function $r_M\colon \mathcal{P}(E(M)) \to \mathbb{R}$ such that

$$r_M(X) = \max\{|Y|\colon Y \subset X, Y \in \mathcal{I}(M)\}.$$

Example

In the graph below, define M as its graphic matroid. Then, $r_M(E(M)) = 4$.



Rank

Theorem

Suppose E is a finite set, and let r: $\mathcal{P}(E(M)) \to \mathbb{Z}$ be a function such that:

- for all $X \subset E$, $0 \leq r(X) \leq |X|$,
- for all $X, Y \subset E$, if $X \subset Y$ then $r(X) \leq r(Y)$,
- and for all $X, Y \subset E$, then $r(X \cup Y) \leq r(X) + r(Y)$.

Then,

$$\mathcal{I} = \{Y \subset E \colon |Y| = r(Y)\}$$

defines an independence system with $E(M) = E$ and $r_M = r$.

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Matroid Rank

Theorem

Suppose E is a finite set, and let r: $\mathcal{P}(E(M)) \to \mathbb{Z}$ be a function such that:

- for all $X \subset E$, $0 \leq r(X) \leq |X|$,
- for all $X, Y \subset E$, if $X \subset Y$ then $r(X) \leq r(Y)$,
- and for all $X, Y \subset E$, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Then,

$$\mathcal{I} = \{Y \subset E \colon |Y| = r(Y)\}$$

defines a matroid with $E(M) = E$ and $r_M = r$.

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Rank Theorem Example

Example

Suppose $E = \{v_1, v_2, v_3\}$ such that:

$$r(\emptyset) = 0, \quad r(\{v_1\}) = r(\{v_2\}) = r(\{v_3\}) = 1,$$

and

$$r(\{v_1, v_2\}) = r(\{v_2, v_3\}) = r(\{v_1, v_3\}) = r(\{v_1, v_2, v_3\}) = 2.$$

Then, this is a matroid since the linear matroid of

$$v_1 = \langle 1, 0\rangle, v_2 = \langle 0, 1\rangle, v_3 = \langle 1, 1\rangle$$

corresponds to this.

Greedy Algorithm

Greedy Algorithm

Greedy Algorithm

Let M be any matroid. Define the greedy algorithm as follows:

1. Define $S_0 = \emptyset, k = 1, U = E(M)$.
2. While $U \neq \emptyset$:
   1. choose $s_k \in U$ of maximum weight and then redefine $U = U - s_k$
   2. if $S_{k-1} + s_k \in \mathcal{I}(M)$, then $S_k = S_{k-1} + s_k$ and redefine $k = k + 1$.

Essentially, the greedy algorithm grabs the largest weight element available while maintaining independence.

When Does the Greedy Algorithm Work?

Theorem

Let M be any matroid, and let r_M be its rank function. The Greedy Algorithm finds maximum-weight independent sets of cardinality k for every k satisfying $1 \leq k \leq r_M(E(M))$.

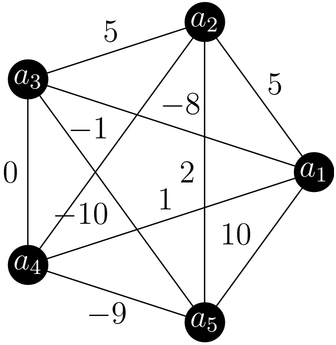
The proof relies on the exchange axiom.

Example 1: Maximum Spanning Tree

Consider the following problem:

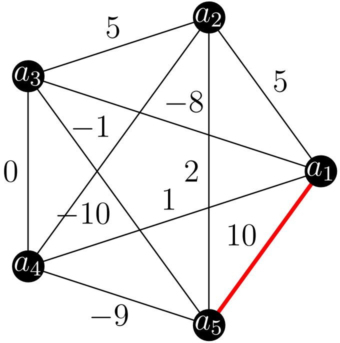
Example

Determine the maximum-weight spanning tree of:



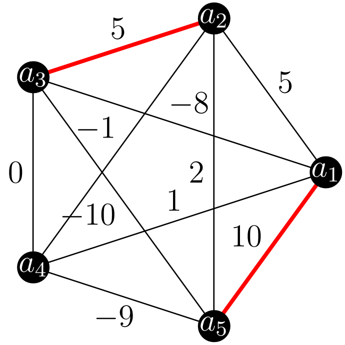
Example 1: Maximum Spanning Tree (cont.)

Example



Example 1: Maximum Spanning Tree (cont.)

Example



Example 1: Maximum Spanning Tree (cont.)

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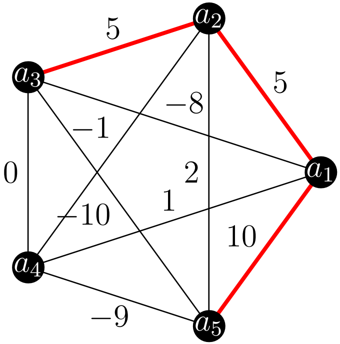

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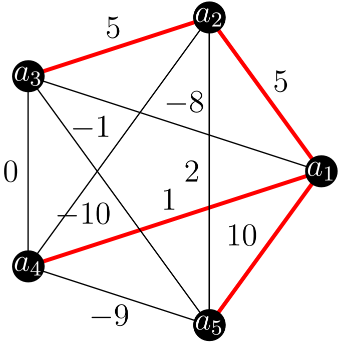

Example 2: Scheduling

Example

Suppose we had n jobs and each takes 1 hour. Job i has a deadline d_i and profit c_i. We wish to maximize profit.

Lemma

Let $E(M) := \{1, \dots, n\}$*. Let* $\mathcal{I}(M) := \{J \subset E(M) : \textit{the jobs in } J \textit{ are completed in time}\}$*. Then, M is a matroid.*

We wish to find the independent set with the largest $\sum c_i$.

Example 2: Scheduling (cont.)

Example

Suppose we start working at 12:00 P.M. We have one processor.

Job j	c_j	d_j
1	20	3:00 P.M.
2	15	1:00 P.M.
3	10	2:00 P.M.
4	10	1:00 P.M.
5	6	2:00 P.M.
6	4	5:00 P.M.
7	3	5:00 P.M.
8	2	4:00 P.M.
9	2	2:00 P.M.
10	1	6:00 P.M.

Example 2: Scheduling (cont.)

Example

Currently 1:00 P.M. We start the highest profit task with $d_j \leq$ 2:00 P.M.

Job j	c_j	d_j
1	20	3:00 P.M.
2	15	1:00 P.M.
3	10	2:00 P.M.
4	10	1:00 P.M.
5	6	2:00 P.M.
6	4	5:00 P.M.
7	3	5:00 P.M.
8	2	4:00 P.M.
9	2	2:00 P.M.
10	1	6:00 P.M.

Example 2: Scheduling (cont.)

Example

Currently 2:00 P.M. We start the highest profit task with $d_j \leq$ 3:00 P.M.

Job j	c_j	d_j
1	20	3:00 P.M.
2	15	1:00 P.M.
3	10	2:00 P.M.
4	10	1:00 P.M.
5	6	2:00 P.M.
6	4	5:00 P.M.
7	3	5:00 P.M.
8	2	4:00 P.M.
9	2	2:00 P.M.
10	1	6:00 P.M.

Example 2: Scheduling (cont.)

Example

Currently 3:00 P.M. We start the highest profit task with $d_j \leq$ 4:00 P.M.

Job j	c_j	d_j
1	20	3:00 P.M.
2	15	1:00 P.M.
3	10	2:00 P.M.
4	10	1:00 P.M.
5	6	2:00 P.M.
6	4	5:00 P.M.
7	3	5:00 P.M.
8	2	4:00 P.M.
9	2	2:00 P.M.
10	1	6:00 P.M.

Example 2: Scheduling (cont.)

Example

Currently 4:00 P.M. We start the highest profit task with $d_j \leq$ 5:00 P.M.

Job j	c_j	d_j
1	20	3:00 P.M.
2	15	1:00 P.M.
3	10	2:00 P.M.
4	10	1:00 P.M.
5	6	2:00 P.M.
6	4	5:00 P.M.
7	3	5:00 P.M.
8	2	4:00 P.M.
9	2	2:00 P.M.
10	1	6:00 P.M.

Example 2: Scheduling (cont.)

Example

Currently 5:00 P.M. We are done. Total profit: 38.

Job j	c_j	d_j
1	20	3:00 P.M.
2	15	1:00 P.M.
3	10	2:00 P.M.
4	10	1:00 P.M.
5	6	2:00 P.M.
6	4	5:00 P.M.
7	3	5:00 P.M.
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When Does the Greedy Algorithm Not Work?

The proof of the greedy algorithm relies on the exchange axiom. The algorithm does not work for all independence systems.

In fact, we can prove the converse of the Greedy Algorithm:

Theorem

Let M be an independence system. If the Greedy Algorithm produces maximum-weight independent sets of all cardinalities for every (nonnegative) weight function, then M is a matroid.

Combining this with our result that the Greedy Algorithm works on matroids:

Corollary

Let M be an independence system. The Greedy Algorithm works if and only if M is a matroid.

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Sketch of Proof

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1. Suppose FTSOC $\exists X, Y \in \mathcal{I}(M)$ with $|X| < |Y|$ and they violate the exchange axiom.
2. Let $c(e) := \begin{cases} 1+\epsilon & \text{if } e \in X \\ 1 & \text{if } e \in Y \setminus X \\ 0 & \text{if } e \in E(M) \setminus (X \cup Y) \end{cases}$ for some $\epsilon > 0$.
3. This specific weight function will always cause the greedy function to not work.

Applications of Greedy Algorithm

It is notoriously difficult to prove the optimality of the greedy algorithm for many problems. However, as long as we can discover a matroid corresponding to a combinatorial problem, we can always execute the Greedy Algorithm.

The greedy algorithm tends to be much simpler and more efficient than most algorithms, as illustrated by our two examples.

Other examples of applications of the greedy algorithm include Dijkstra's shortest path algorithm, decision-tree learning, and data compression.

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J. Lee, *A First Course in Combinatorial Optimization.*