Edge bounds on k-visibility hypergraphs

Alexander Wang
Mentor: Dr. Jesse Geneson

October 19, 2025
MIT PRIMES Conference

Table of Contents

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.
- Chips can be modelled as visibility graphs:

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.
- Chips can be modelled as visibility graphs:
  - Macrocells are vertices

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.
- Chips can be modelled as visibility graphs:
  - Macrocells are vertices
  - Connections between macrocells are edges

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.
- Chips can be modelled as visibility graphs:
  - Macrocells are vertices
  - Connections between macrocells are edges
- Certain design constraints limit the possible connections

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.
- Chips can be modelled as visibility graphs:
  - Macrocells are vertices
  - Connections between macrocells are edges
- Certain design constraints limit the possible connections
- What is the maximum number of connections between some number of macrocells? What is the optimal layout?

Motivation

- Very-large-scale integration (VLSI) is the technology used to place millions or billions of transistors onto a single chip. These chips are used everywhere, including in computers and phones, so they must be designed efficiently.
- Chips can be modelled as visibility graphs:
  - Macrocells are vertices
  - Connections between macrocells are edges
- Certain design constraints limit the possible connections
- What is the maximum number of connections between some number of macrocells? What is the optimal layout?
- Answering these questions can reduce the size, number of layers, and complexity, leading to more compact and efficient design

Visibility graph

- A *visibility graph* is a graph where vertices and edges are represented in the plane with certain conditions

Visibility graph

- A *visibility graph* is a graph where vertices and edges are represented in the plane with certain conditions
  - The vertices are represented by regions on the plane

Visibility graph

- A *visibility graph* is a graph where vertices and edges are represented in the plane with certain conditions
  - The vertices are represented by regions on the plane
  - Two vertices are connected by an edge if they are visible to each other

Visibility graph

- A *visibility graph* is a graph where vertices and edges are represented in the plane with certain conditions
  - The vertices are represented by regions on the plane
  - Two vertices are connected by an edge if they are visible to each other
- Different types of visibility graphs can be formed by specifying the definition of visibility

Visibility graph

- A *visibility graph* is a graph where vertices and edges are represented in the plane with certain conditions
  - The vertices are represented by regions on the plane
  - Two vertices are connected by an edge if they are visible to each other
- Different types of visibility graphs can be formed by specifying the definition of visibility
  - Example: Two regions are visible to each other if there is a line segment connecting the two regions which does not intersect any other region

Visibility graph

- A *visibility graph* is a graph where vertices and edges are represented in the plane with certain conditions
  - The vertices are represented by regions on the plane
  - Two vertices are connected by an edge if they are visible to each other
- Different types of visibility graphs can be formed by specifying the definition of visibility
  - Example: Two regions are visible to each other if there is a line segment connecting the two regions which does not intersect any other region

Question

What is the maximum number of edges in a visibility graph with a fixed number of vertices?

- We will be answering this question for bar, semi-bar, arc, semi-arc, and rectangle visibility graphs, k-visibility graphs, and k-visibility hypergraphs.

Bar visibility graph

- Draw some horizontal line segments, called *bars*. These are the vertices of the graph.

Bar visibility graph

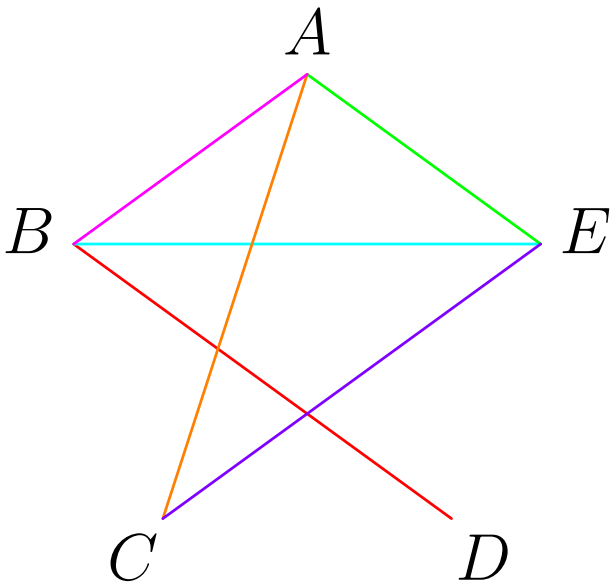
- Draw some horizontal line segments, called *bars*. These are the vertices of the graph.
- Draw vertical line segments that have endpoints on two bars while not passing through any other bar. These are the edges of the graph.

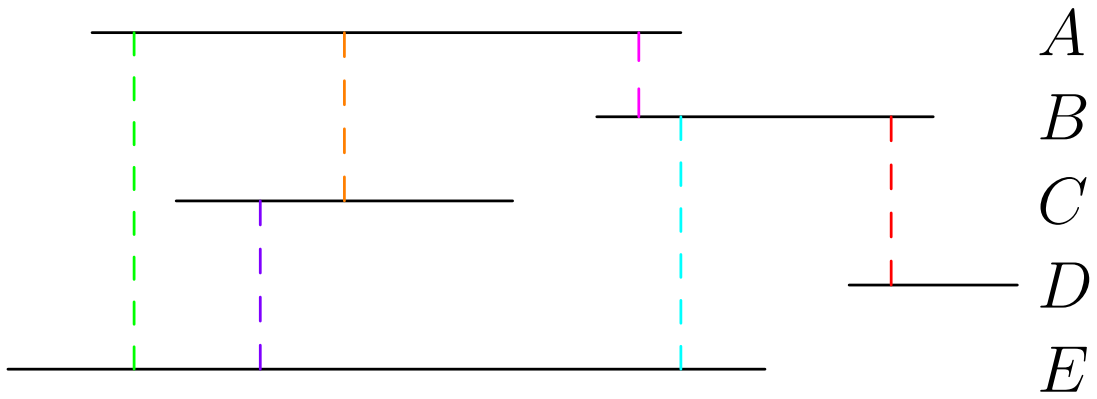
Bar visibility graph

- Draw some horizontal line segments, called *bars*. These are the vertices of the graph.
- Draw vertical line segments that have endpoints on two bars while not passing through any other bar. These are the edges of the graph.
- This graph is a *bar visibility graph*.

Bar visibility graph

- Draw some horizontal line segments, called *bars*. These are the vertices of the graph.
- Draw vertical line segments that have endpoints on two bars while not passing through any other bar. These are the edges of the graph.
- This graph is a *bar visibility graph*.





Bar visibility graph

- What is the maximum number of edges in a bar visibility graph with n vertices?

Bar visibility graph

- What is the maximum number of edges in a bar visibility graph with n vertices?

Theorem

Every bar visibility graph is planar.

Bar visibility graph

- What is the maximum number of edges in a bar visibility graph with n vertices?

Theorem

Every bar visibility graph is planar.

- Therefore, each bar visibility graph with $n \geq 3$ vertices has at most $3n - 6$ edges.

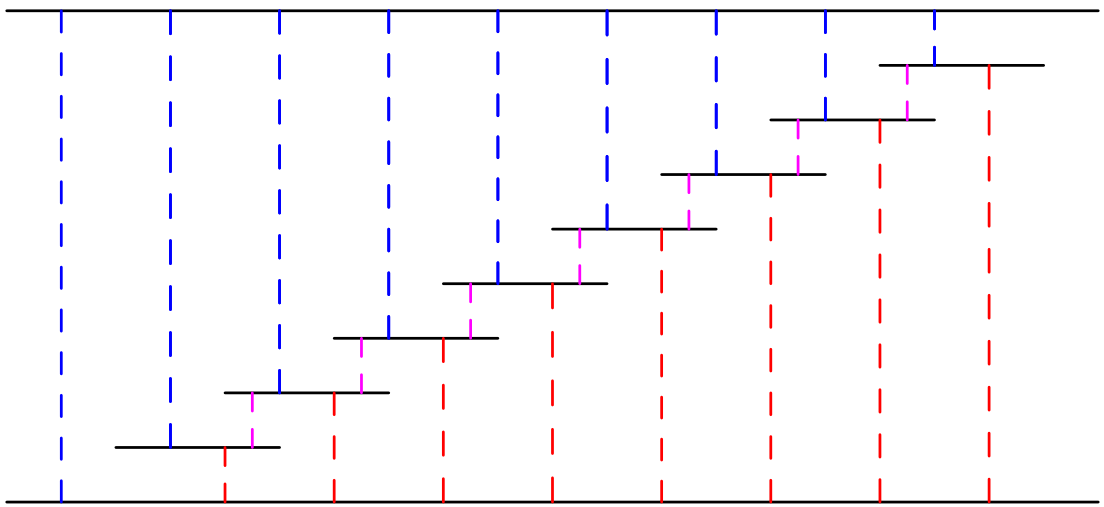
Bar visibility graph

- What is the maximum number of edges in a bar visibility graph with n vertices?

Theorem

Every bar visibility graph is planar.

- Therefore, each bar visibility graph with $n \geq 3$ vertices has at most $3n - 6$ edges.
- Equality holds.

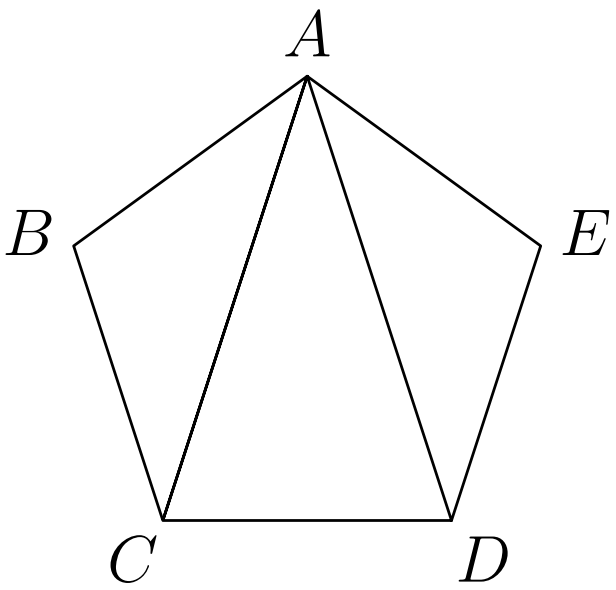


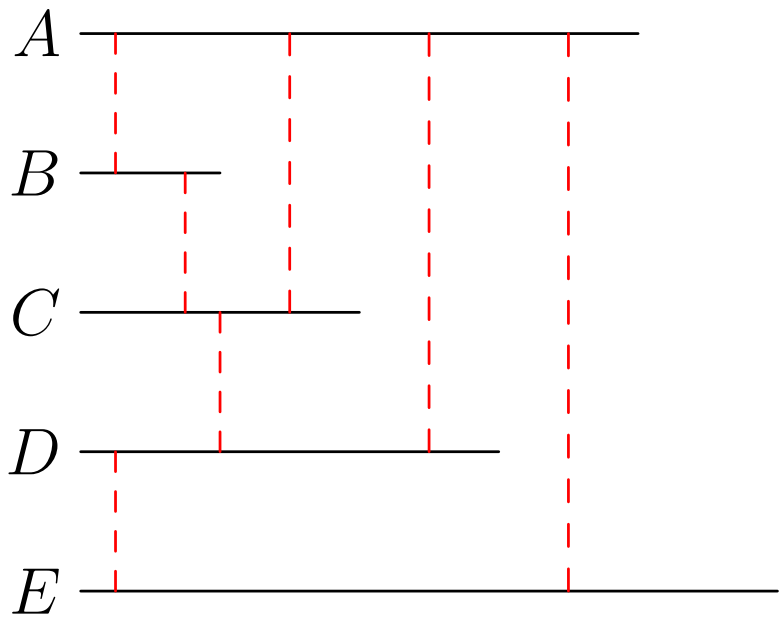
Semi-bar visibility graph

- A *semi-bar visibility graph* is a bar visibility graph with the additional condition that the left endpoint of each bar is aligned at the same x-coordinate.

Semi-bar visibility graph

- A *semi-bar visibility graph* is a bar visibility graph with the additional condition that the left endpoint of each bar is aligned at the same x-coordinate.





Arc visibility graph

- Draw arcs centered at the origin. These are the vertices of the graph.

Arc visibility graph

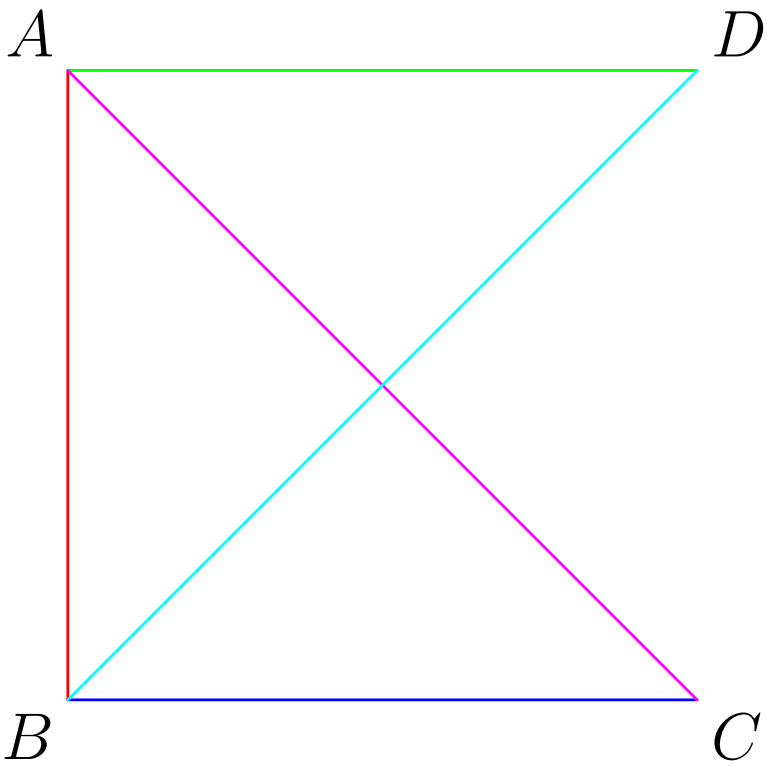
- Draw arcs centered at the origin. These are the vertices of the graph.
- Draw line segments which have endpoints collinear with the origin. This line segment can pass through the origin but does not have to pass through the origin. These line segments are the edges of the graph.

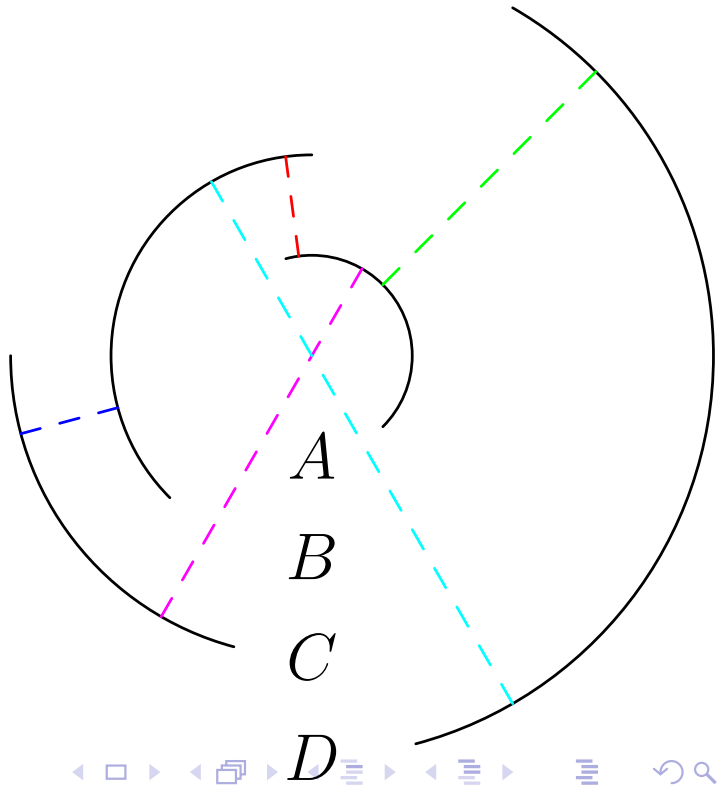
Arc visibility graph

- Draw arcs centered at the origin. These are the vertices of the graph.
- Draw line segments which have endpoints collinear with the origin. This line segment can pass through the origin but does not have to pass through the origin. These line segments are the edges of the graph.
- This graph is an *arc visibility graph*.

Arc visibility graph

- Draw arcs centered at the origin. These are the vertices of the graph.
- Draw line segments which have endpoints collinear with the origin. This line segment can pass through the origin but does not have to pass through the origin. These line segments are the edges of the graph.
- This graph is an *arc visibility graph*.



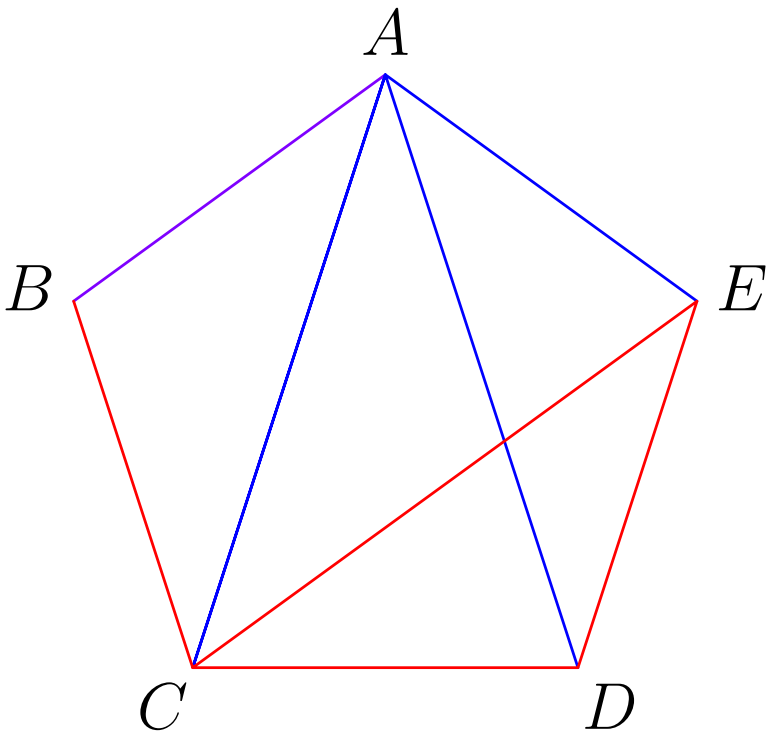


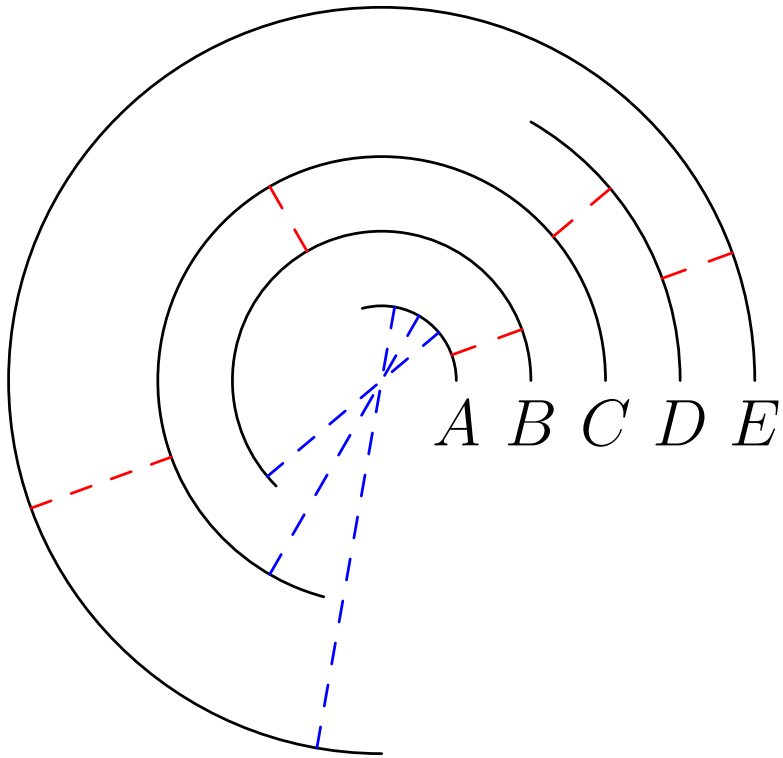
Semi-arc visibility graph

- A *semi-arc visibility graph* is an arc visibility graph where each arc must extend counterclockwise starting from the positive x-axis.

Semi-arc visibility graph

- A *semi-arc visibility graph* is an arc visibility graph where each arc must extend counterclockwise starting from the positive x-axis.





Rectangle visibility graph

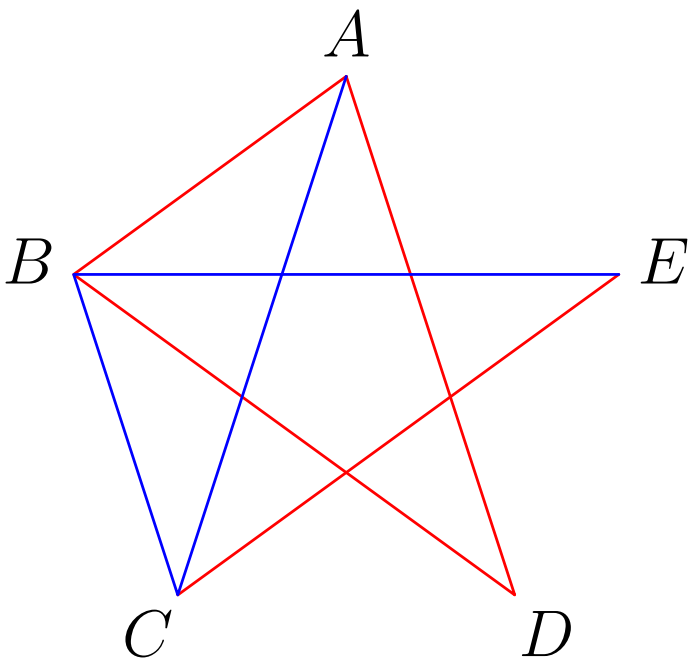
- Draw rectangles with sides parallel to the x-axis and y-axis. These are the vertices of the graph.

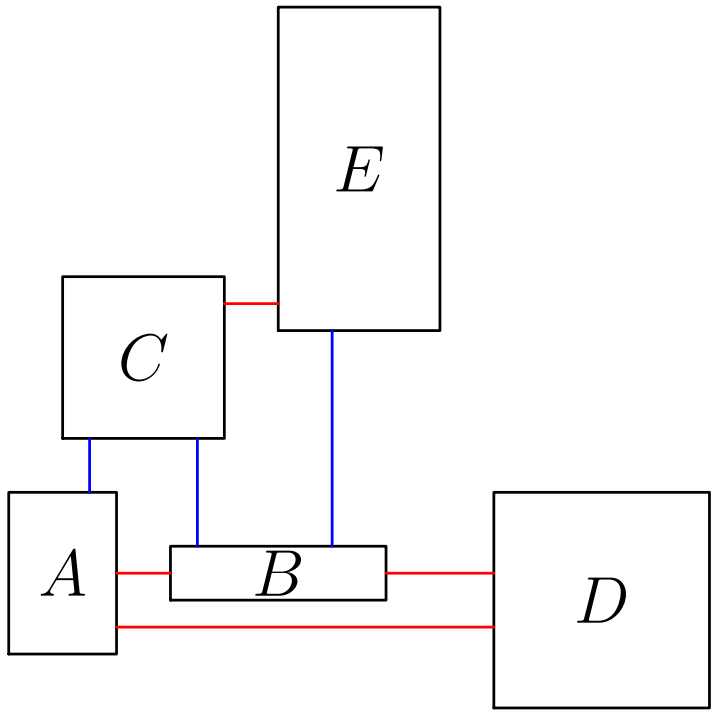
Rectangle visibility graph

- Draw rectangles with sides parallel to the x-axis and y-axis. These are the vertices of the graph.
- Draw horizontal or vertical line segments which connect two rectangles and do not intersect any other rectangle. These are the edges of the graph.

Rectangle visibility graph

- Draw rectangles with sides parallel to the x-axis and y-axis. These are the vertices of the graph.
- Draw horizontal or vertical line segments which connect two rectangles and do not intersect any other rectangle. These are the edges of the graph.
- This graph is a *rectangle visibility graph*.





k-visibility graphs

- A k-visibility graph is a generalization of a visibility graph.

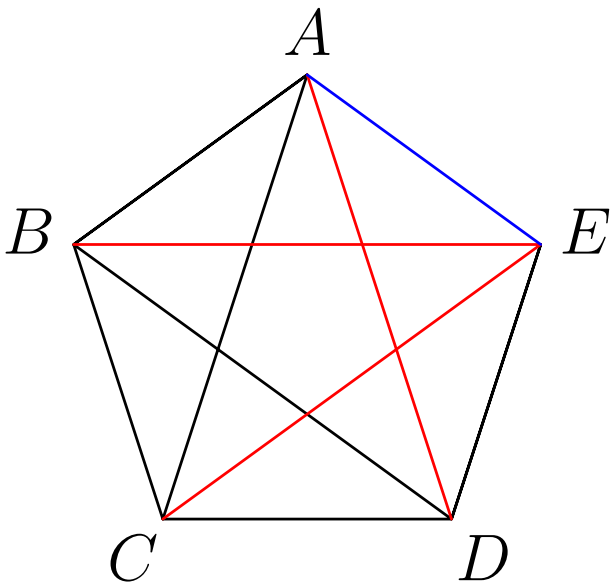
k-visibility graphs

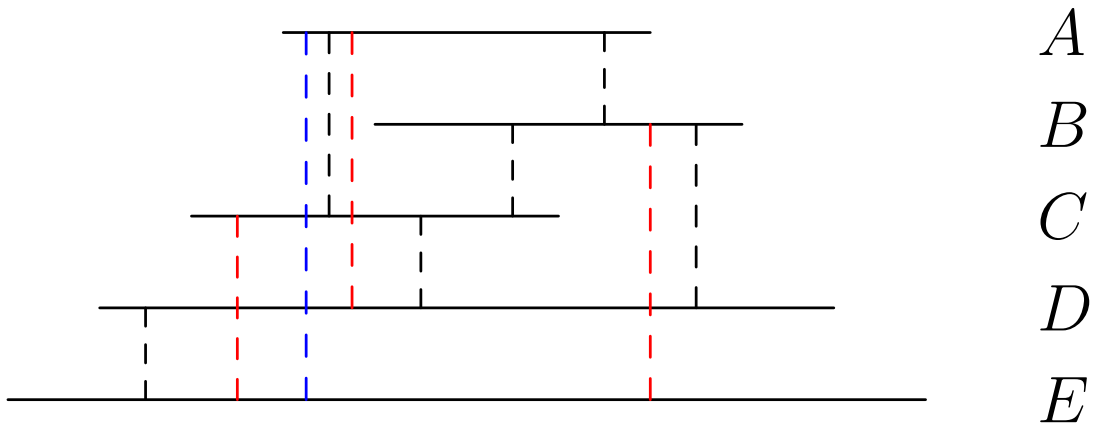
- A k-visibility graph is a generalization of a visibility graph.
- In a visibility graph, each edge is a line segment that connects two vertices which does not intersect any other vertex. In a k-visibility graph, this condition is modified so that each edge connecting two vertices can intersect at most k other vertices.

k-visibility graphs

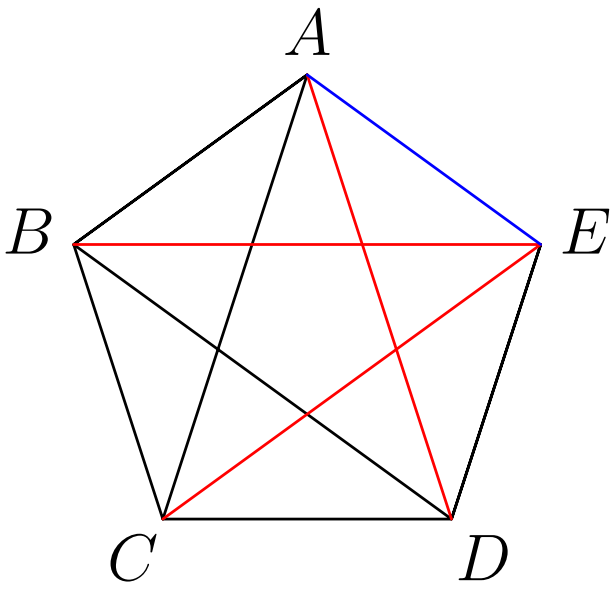
- A k-visibility graph is a generalization of a visibility graph.
- In a visibility graph, each edge is a line segment that connects two vertices which does not intersect any other vertex. In a k-visibility graph, this condition is modified so that each edge connecting two vertices can intersect at most k other vertices.
- A 0-visibility graph is the same as the corresponding visibility graph.

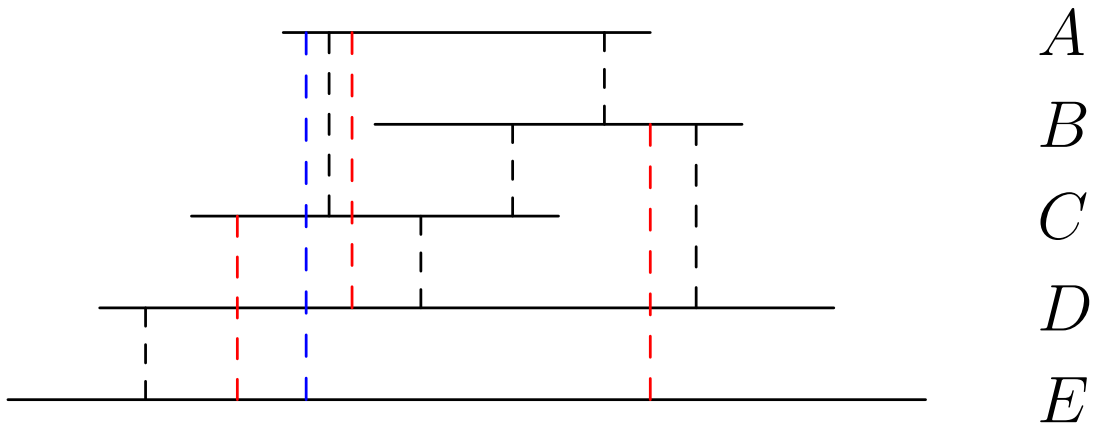
Example: bar k-visibility graph





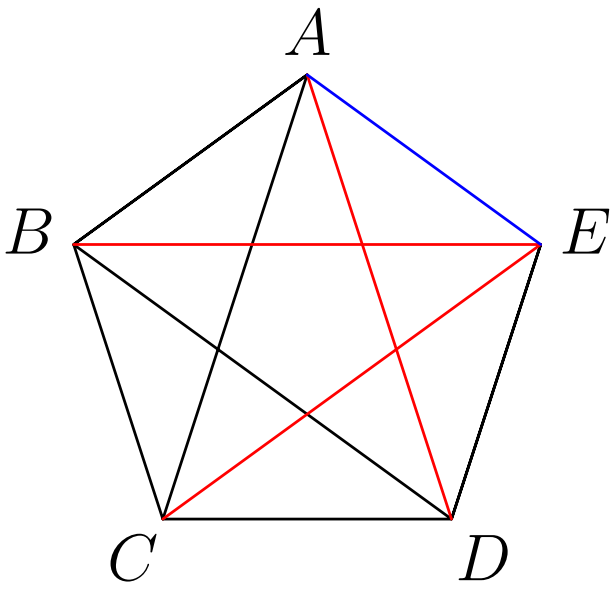
Example: bar k-visibility graph

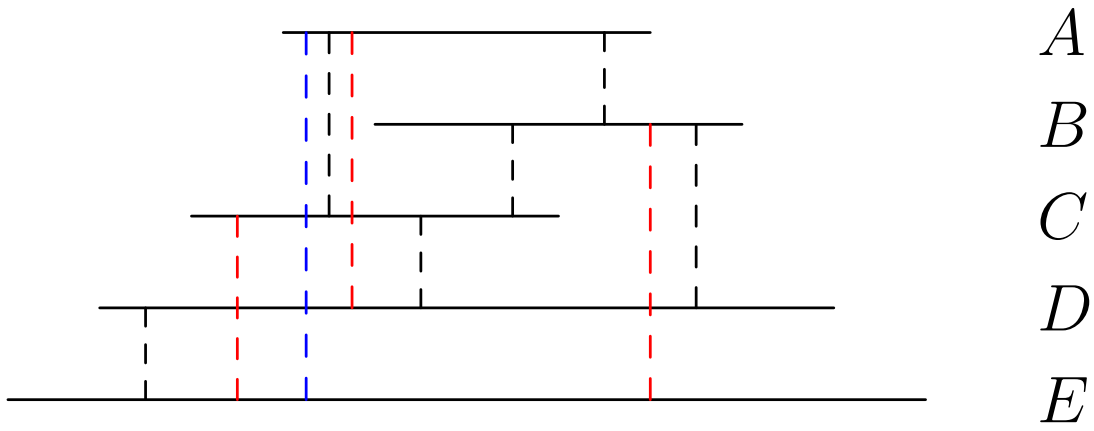




- The bar 0-visibility graph contains the black edges.

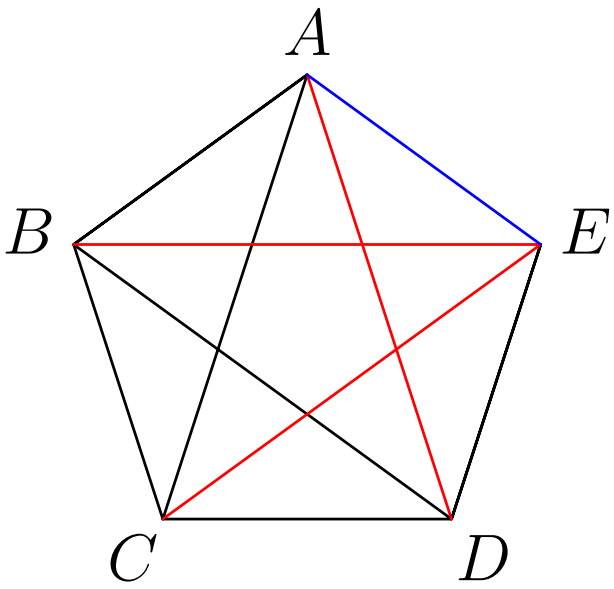
Example: bar k-visibility graph

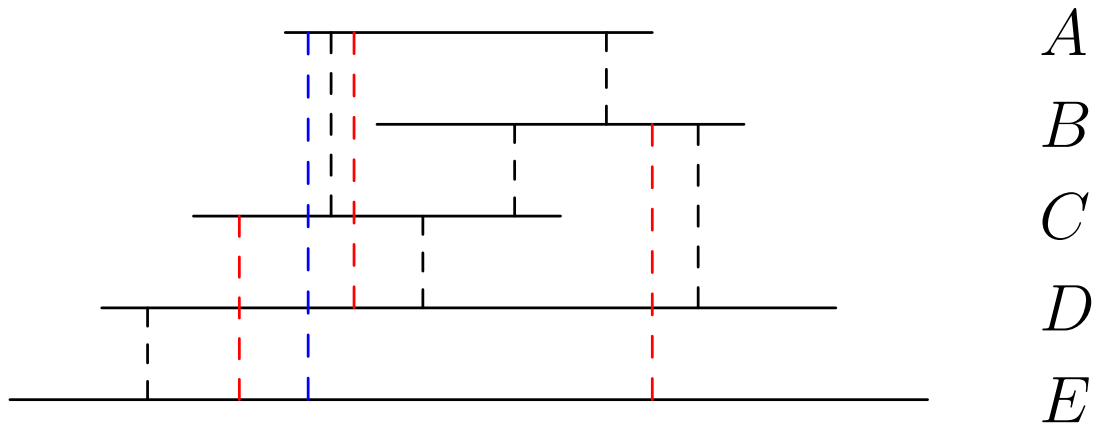




- The bar 0-visibility graph contains the black edges.
- The bar 1-visibility graph contains the black and red edges.

Example: bar k-visibility graph





- The bar 0-visibility graph contains the black edges.
- The bar 1-visibility graph contains the black and red edges.
- The bar 2-visibility graph contains the black, red, and blue edges.

Edge bounds in k-visibility graphs

Maximum number of edges in a k-visibility graph with n vertices:

Edge bounds in k-visibility graphs

Maximum number of edges in a k-visibility graph with n vertices:

Theorem (Hartke et al, 2007)

The maximum number of edges in a bar k-visibility graph with n vertices is $(k+1)(3n-4k-6)$ when $n \geq 4k+4$.

Edge bounds in k-visibility graphs

Maximum number of edges in a k-visibility graph with n vertices:

Theorem (Hartke et al, 2007)

The maximum number of edges in a bar k-visibility graph with n vertices is $(k+1)(3n-4k-6)$ when $n \geq 4k+4$.

Theorem (Felsner and Massow, 2008)

The maximum number of edges in a semi-bar k-visibility graph with n vertices is $(k+1)(2n-2k-3)$ when $n \geq 2k+3$ and $\binom{n}{2}$ when $n \leq 2k+2$.

Edge bounds in k-visibility graphs

Maximum number of edges in a k-visibility graph with n vertices:

Theorem (Hartke et al, 2007)

The maximum number of edges in a bar k-visibility graph with n vertices is $(k+1)(3n-4k-6)$ when $n \geq 4k+4$.

Theorem (Felsner and Massow, 2008)

The maximum number of edges in a semi-bar k-visibility graph with n vertices is $(k+1)(2n-2k-3)$ when $n \geq 2k+3$ and $\binom{n}{2}$ when $n \leq 2k+2$.

Theorem (Sawhney and Weed, 2016)

The maximum number of edges in a semi-arc k-visibility graph with n vertices is $(k+1)\left(2n-\frac{k+2}{2}\right)$ when $n \geq 3k+3$.

Edge bounds in k-visibility graphs

Theorem (Sawhney and Weed, 2016)

The maximum number of edges in an arc k-visibility graph is at most $(k+1)\left(3n - \frac{3k+6}{2}\right)$ for $n \geq 4k+5$.

Edge bounds in k-visibility graphs

Theorem (Sawhney and Weed, 2016)

The maximum number of edges in an arc k-visibility graph is at most $(k+1)\left(3n - \frac{3k+6}{2}\right)$ for $n \geq 4k+5$.

Is this upper bound optimal?

Arc k-visibility graphs

We prove that Sawhney and Weed's upper bound for arc k-visibility graphs is optimal when $n \geq 7k + 6$ by providing a construction.

Arc k-visibility graphs

We prove that Sawhney and Weed's upper bound for arc k-visibility graphs is optimal when $n \geq 7k + 6$ by providing a construction.

Theorem

The maximum number of edges in an arc k-visibility hypergraph with n vertices is $(k+1)\left(3n - \frac{3k+6}{2}\right)$ for $n \geq 7k + 6$.

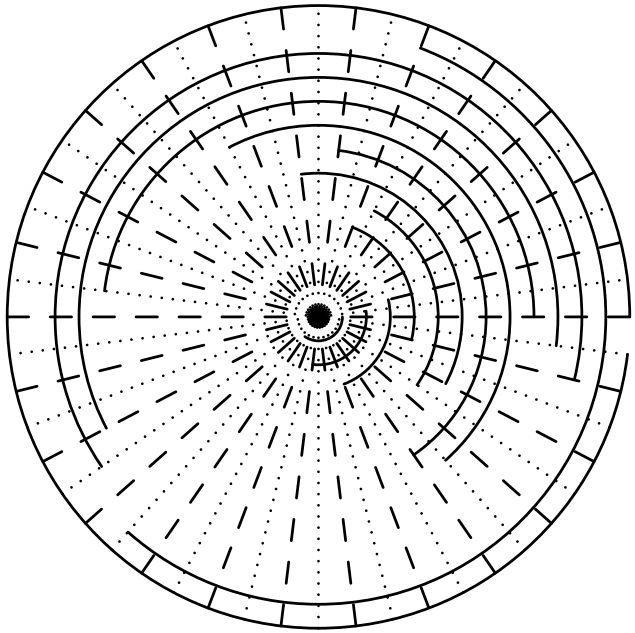
Arc k-visibility graphs

We prove that Sawhney and Weed's upper bound for arc k-visibility graphs is optimal when $n \geq 7k + 6$ by providing a construction.

Theorem

The maximum number of edges in an arc k-visibility hypergraph with n vertices is $(k+1)\left(3n - \frac{3k+6}{2}\right)$ for $n \geq 7k + 6$.

Construction for $n = 13$, $k = 1$:



k-visibility hypergraphs

- A k-visibility hypergraph is another generalization of a visibility graph.

k-visibility hypergraphs

- A k-visibility hypergraph is another generalization of a visibility graph.
- In a k-visibility graph, each segment is allowed to intersect at most $k + 2$ vertices in total.

k-visibility hypergraphs

- A k-visibility hypergraph is another generalization of a visibility graph.
- In a k-visibility graph, each segment is allowed to intersect at most $k+2$ vertices in total.
- In a k-visibility hypergraph, each segment must intersect exactly $k+2$ distinct vertices and an edge now consists of the set of all vertices the segment intersects.

k-visibility hypergraphs

- A k-visibility hypergraph is another generalization of a visibility graph.
- In a k-visibility graph, each segment is allowed to intersect at most $k+2$ vertices in total.
- In a k-visibility hypergraph, each segment must intersect exactly $k+2$ distinct vertices and an edge now consists of the set of all vertices the segment intersects.
- A 0-visibility hypergraph is the same as the corresponding visibility graph.

k-visibility hypergraphs

- A k-visibility hypergraph is another generalization of a visibility graph.
- In a k-visibility graph, each segment is allowed to intersect at most $k+2$ vertices in total.
- In a k-visibility hypergraph, each segment must intersect exactly $k+2$ distinct vertices and an edge now consists of the set of all vertices the segment intersects.
- A 0-visibility hypergraph is the same as the corresponding visibility graph.
- In 2015, Geneson and Shen introduced bar k-visibility hypergraphs to answer a question on the extremal function of matrices. They proved a linear upper bound on maximum number of 1s in a 0–1 matrix which does not contain a certain pattern of 1s by converting the 1s into a bar k-visibility hypergraph.

Bar k-visibility hypergraphs

Theorem

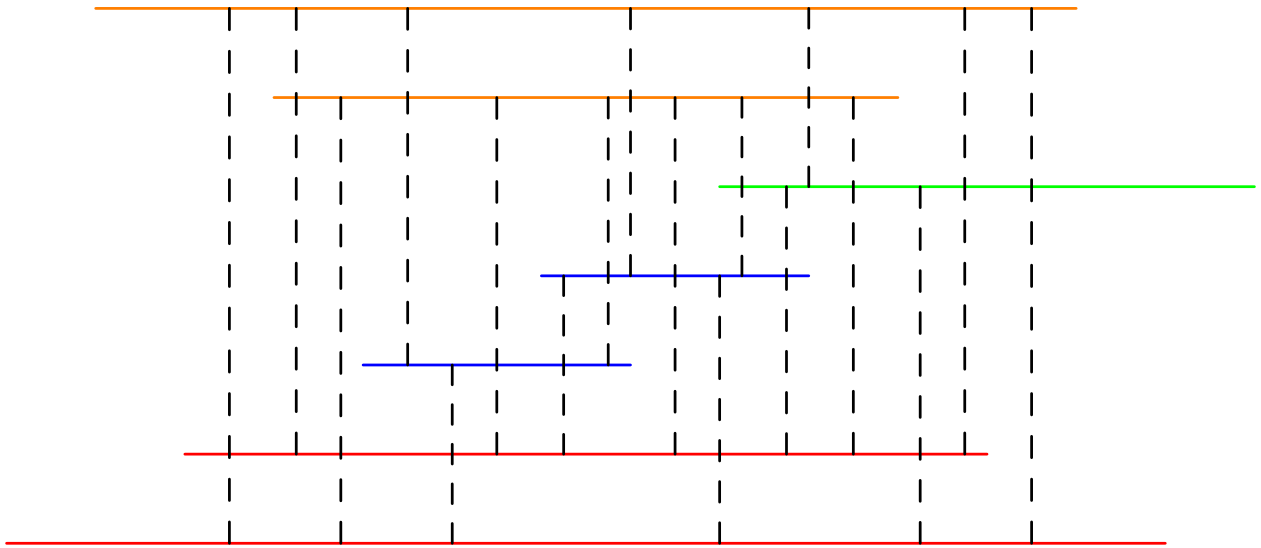
The maximum number of edges in a bar k-visibility hypergraph on n vertices is 0 if $n \leq k+1$, $(n-k-1)^2$ if $k+2 \leq n \leq 2k+2$, and $(2k+3)n - 3k^2 - 8k - 6$ if $n \geq 2k+3$.

Bar k-visibility hypergraphs

Theorem

The maximum number of edges in a bar k-visibility hypergraph on n vertices is 0 if $n \leq k+1$, $(n-k-1)^2$ if $k+2 \leq n \leq 2k+2$, and $(2k+3)n - 3k^2 - 8k - 6$ if $n \geq 2k+3$.

Construction for $n = 7$, $k = 1$:



Semi-bar k-visibility hypergraphs

Theorem

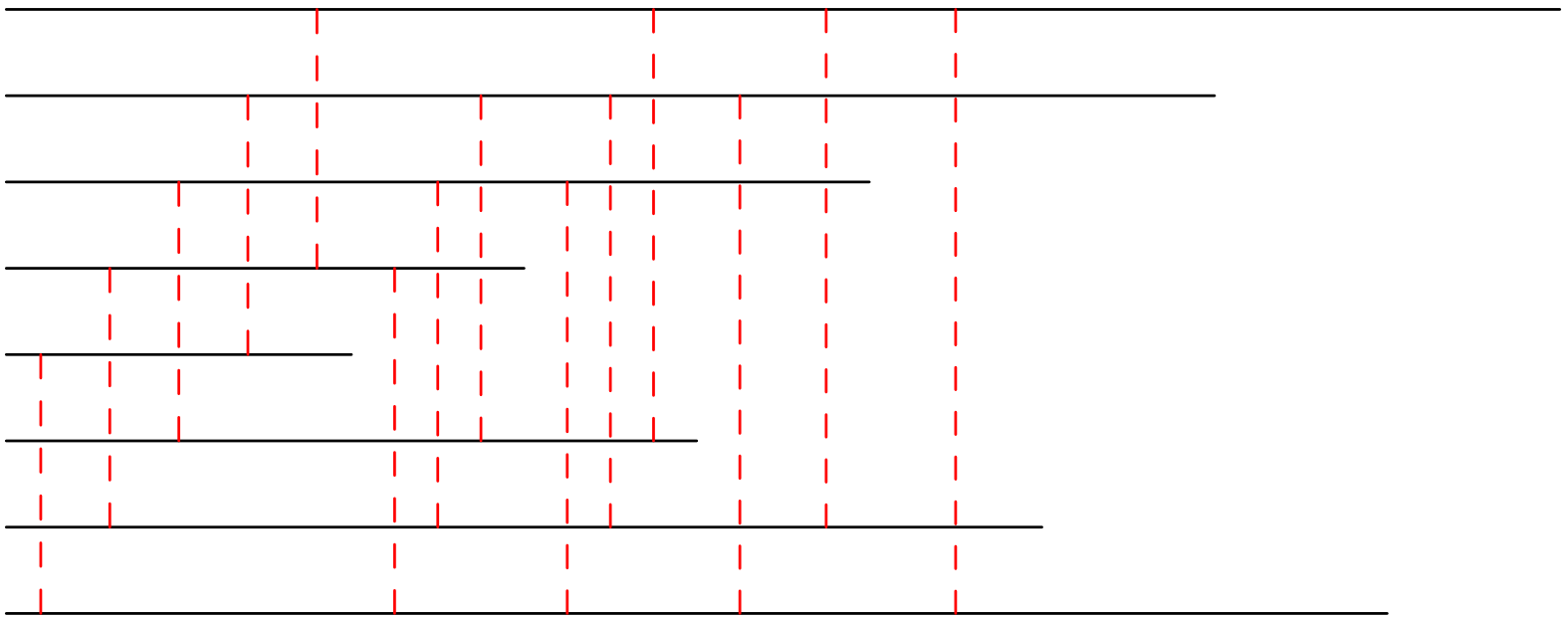
The maximum number of edges in a semi-bar k-visibility hypergraph is 0 when $n \leq k$, $\frac{(n-1-k)(n-k)}{2}$ when $k+1 \leq n \leq 2k+3$, and $\frac{(k+2)(2n-3k-3)}{2}$ when $n \geq 2k+4$.

Semi-bar k-visibility hypergraphs

Theorem

The maximum number of edges in a semi-bar k-visibility hypergraph is 0 when $n \leq k$, $\frac{(n-1-k)(n-k)}{2}$ when $k+1 \leq n \leq 2k+3$, and $\frac{(k+2)(2n-3k-3)}{2}$ when $n \geq 2k+4$.

Construction for $n = 8$, $k = 2$:



Arc k-visibility hypergraphs

Theorem

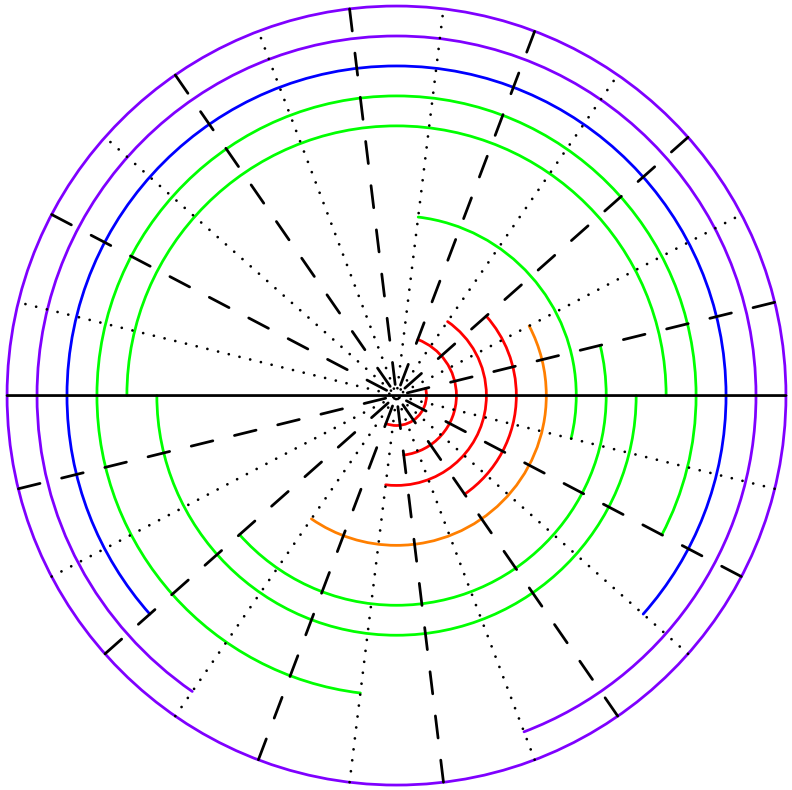
The maximum number of edges in a arc k-visibility hypergraph with n vertices is $n(2k+3) - (k+1)(k+3)$ for $n \geq 2k+5$.

Arc k-visibility hypergraphs

Theorem

The maximum number of edges in a arc k-visibility hypergraph with n vertices is $n(2k+3) - (k+1)(k+3)$ for $n \geq 2k+5$.

Construction for $n = 13$, $k = 4$:



Semi-arc k-visibility hypergraphs

Theorem

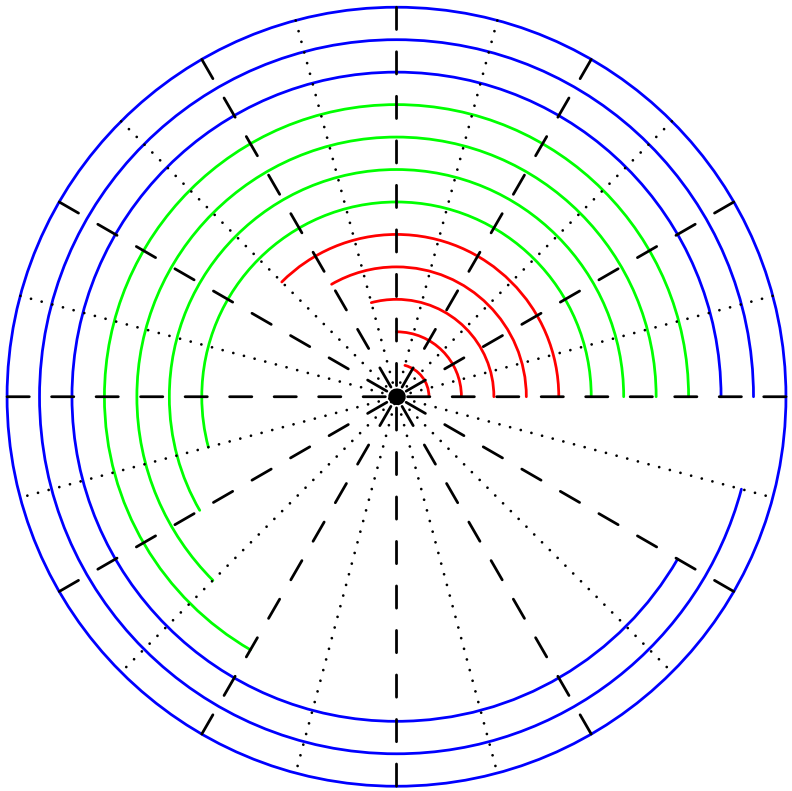
The maximum number of edges in a semi-arc k-visibility hypergraph with n vertices is $n(k+2) - \frac{(k+1)(k+2)}{2}$ for $n \geq 3k+5$.

Semi-arc k-visibility hypergraphs

Theorem

The maximum number of edges in a semi-arc k-visibility hypergraph with n vertices is $n(k+2) - \frac{(k+1)(k+2)}{2}$ for $n \geq 3k+5$.

Construction for $n = 12$, $k = 2$:



Rectangle k-visibility hypergraphs

Theorem

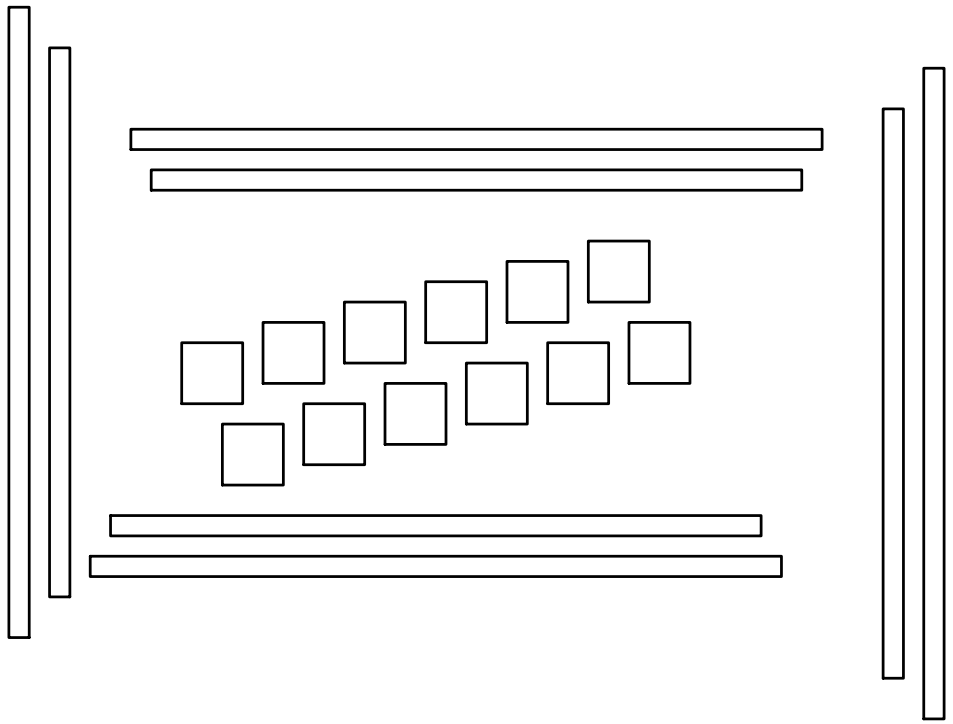
The maximum number of edges in a rectangle k-visibility hypergraph with n vertices is between $(4k+6)n - 4(k+1)(3k+5)$ and $(4k+6)n - (k+1)(9k+11)$.

Rectangle k-visibility hypergraphs

Theorem

The maximum number of edges in a rectangle k-visibility hypergraph with n vertices is between $(4k+6)n - 4(k+1)(3k+5)$ and $(4k+6)n - (k+1)(9k+11)$.

Lower bound construction for $n = 20$, $k = 1$:



Acknowledgements

- I would like to thank my mentor Dr. Jesse Geneson for his invaluable support and guidance throughout my research.

Acknowledgements

- I would like to thank my mentor Dr. Jesse Geneson for his invaluable support and guidance throughout my research.
- I would also like to thank the MIT PRIMES-USA program for providing me with this research opportunity.

Acknowledgements

- I would like to thank my mentor Dr. Jesse Geneson for his invaluable support and guidance throughout my research.
- I would also like to thank the MIT PRIMES-USA program for providing me with this research opportunity.
- Finally, I would like to thank my parents for supporting me through my entire mathematical journey.

References

- A. Dean, W. Evans, E. Gethner, J. Laison, M. Safari, and W. Trotter. Bar k-Visibility Graphs. Journal of Graph Algorithms and Applications 11(1) (2007) 45–59.
- S. Felsner and M. Massow. Parameters of bar k-visibility graphs. Journal of Graph Algorithms and Applications 12(1) (2008), 5-27.
- J. Geneson and L. Shen. Linear bounds on matrix extremal functions using visibility hypergraphs. Discrete Mathematics 338 (2015), 2437-2441.
- S. Hartke, J. Vandenbussche, and P. Wenger. Further results on bar k-visibility graphs. SIAM Journal on Discrete Mathematics 21(2) (2007) 523-531.
- M. Sawhney and J. Weed. Further results on arc and bar k-visibility graphs. arXiv:1601.01231 [math.CO] (2016).