18.354 – Nonlinear Dynamics II: Continuum systems
Problem Set 4: Surface Tension & Elasticity
Due: Monday, April 7 (by 3pm in class)

Problem 1: Capillary rise

Consider an infinite pool of liquid, with surface tension γ at the air interface. You know from common experience that if you put a vertical wall into the pool, the liquid will climb up the wall and forms a meniscus. The goal of this problem is to calculate how much it moves up the wall. Suppose the liquid has an angle of contact θ_c with the solid wall. Use the calculus of variations to determine $h(x)$, the shape of the interface that minimises the energy of the system (*hint: you need to consider the contribution to the energy from both surface tension and the gravitational potential*).

This problem can be solved both analytically and numerically, and you should try both. Write your total energy as in integral in x. Can you use the special form of Euler-Lagrange? You should obtain a first order ODE for $h(x)$. To find the analytic solution, you should solve this by separation of variables (you may need to look up the integral for h). Your boundary conditions are given to you physically as (i) the contact angle on the solid wall ($h'(0) = \cot\theta$), and (ii) the height goes to the resting height of the liquid far from the wall. To solve the problem numerically, you can use the ODE to relate $h'(0)$ to $h(0)$. Pick a value of θ_c, and, then use an ODE solver in MATLAB (either ode45 or ode15s) to find $h(x)$. Is the condition on $h(x)$ far away from the wall satisfied? You may also want to clean up the problem by scaling h and x by the capillary length $L_c = \sqrt{\gamma/\rho g}$. Compare your numerical solution with the analytical one.

Problem 2: Linear Elasticity and Einstein Notation

Using Einstein notation, the most general expression for the free energy of a deformed isotropic body is

$$E = \frac{1}{2}\lambda e_{ii}^2 + \mu e_{ik}^2,$$

where λ and μ are called Lamè coefficients. It is convenient to replace this by another formula, decomposing the energy into a pure shear and a pure compression. Then E becomes

$$E = \mu(e_{ik} - \frac{1}{3}\delta_{ik}e_{ll})^2 + \frac{1}{2}Ke_{ll}^2.$$

This is the expression we used in class, where K and μ are respectively the modulus of compression and rigidity. Rewrite the above expressions, including the summation signs where appropriate, writing your steps explicitly and clearly.

(a) Find an expression for K in terms of λ and μ.

(b) The stress tensor σ_{ik} is related to the free energy via $\sigma_{ik} = \partial E/\partial e_{ik}$. Show that

$$\sigma_{ik} = K e_{ll}\delta_{ik} + 2\mu(e_{ik} - \frac{1}{3}\delta_{ik}e_{ll}).$$

(c) Show that the stress tensor, e_{ij}, can be determined by inverting the expression for the stress tension, σ_{ik}, that you found above, such that,

$$e_{ij} = \frac{\sigma_{ll}}{9K}\delta_{ij} + \frac{\sigma_{ij} - \frac{1}{3}\sigma_{ll}\delta_{ij}}{2\mu}. \tag{1}$$

Having done this, now rewrite your steps for (a) and (b) in concise form, using Einstein notation.

Hint - For (b) you will find the following relations helpful:

$$\begin{aligned}
\frac{\partial e_{ik}}{\partial e_{mn}} &= \delta_{im}\delta_{kn} \\
\frac{\partial(e_{ll})}{\partial e_{mn}} &= \frac{\partial(e_{11} + e_{22} + e_{33})}{\partial e_{mn}} = \delta_{mn} \\
\delta_{ik}^2 &= \sum_{i,k=1}^{3} \delta_{ik}\delta_{ik} = \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = 3
\end{aligned}$$