

Brownian motion

(cont.)

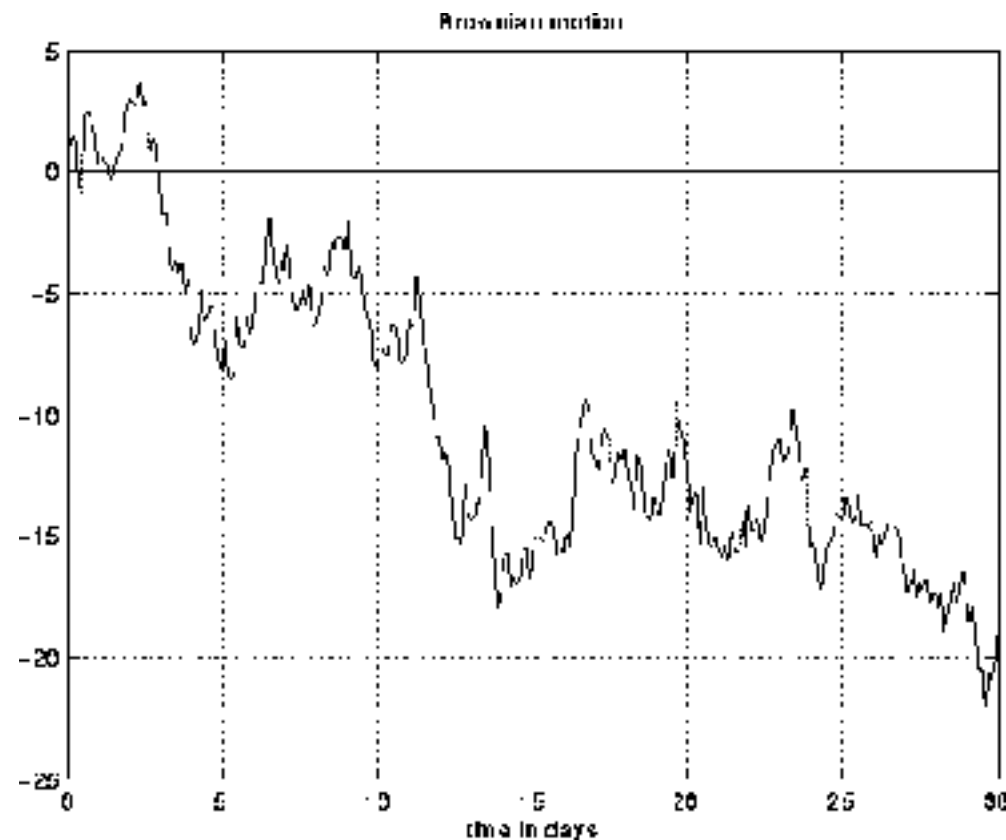
18.S995 - L05

1.2 Brownian motion

Diffusion equation with constant drift

$$\partial_t p = -u \partial_x p + D \partial_{xx} p \quad (1.19a)$$

Path-wise representation of typical trajectories ?



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1.2.1 SDEs and discretization rules

The continuous stochastic process $X(t)$ described by Eq. (1.19a) or, equivalently, Eq. (1.20) can also be represented by the stochastic differential equation

$$dX(t) = u dt + \sqrt{2D} dB(t). \quad (1.25)$$



Wiener process

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Here, $dX(t) = X(t + dt) - X(t)$ is increment of the stochastic particle trajectory $X(t)$, whilst $dB(t) = B(t + dt) - B(t)$ denotes an increment of the standard Brownian motion (or Wiener) process $B(t)$, uniquely defined by the following properties³:

- (i) $B(0) = 0$ with probability 1.



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- (i) $B(0) = 0$ with probability 1.
- (ii) $B(t)$ is stationary, i.e., for $t > s \geq 0$ the increment $B(t) - B(s)$ has the same distribution as $B(t - s)$.



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- (ii) $B(t)$ is stationary, i.e., for $t > s \geq 0$ the increment $B(t) - B(s)$ has the same distribution as $B(t - s)$.
- (iii) $B(t)$ has independent increments. That is, for all $t_n > t_{n-1} > \dots > t_2 > t_1$, the random variables $B(t_n) - B(t_{n-1}), \dots, B(t_2) - B(t_1), B(t_1)$ are independently distributed (i.e., their joint distribution factorizes).



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- (iv) $B(t)$ has Gaussian distribution with variance t for all $t \in (0, \infty)$.



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- (iv) $B(t)$ has Gaussian distribution with variance t for all $t \in (0, \infty)$.
- (v) $B(t)$ is continuous with probability 1.

The probability distribution \mathbb{P} governing the driving process $B(t)$ is commonly known as the Wiener measure.

SDEs in physicist's notation

$$dX(t) = u dt + \sqrt{2D} dB(t). \quad (1.25)$$

Although the derivative $\xi(t) = dB/dt$ is not well-defined mathematically, Eq. (1.25) is in the physics literature often written in the form

$$\dot{X}(t) = u + \sqrt{2D} \xi(t). \quad (1.26)$$

The random driving function $\xi(t)$ is then referred to as Gaussian white noise, characterized by

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(s) \rangle = \delta(t - s), \quad (1.27)$$

with $\langle \cdot \rangle$ denoting an average with respect to the Wiener measure.

Stochastic differential calculus

$$dX(t) = u \, dt + \sqrt{2D} \, dB(t). \quad (1.25)$$

Ito's formula Note that property (iv) implies that $\mathbb{E}[dB^2] = dt$. This justifies the following heuristic derivation of Ito's formula for the differential change of some real-valued function $F(x)$

$$\begin{aligned} dF(X(t)) &:= F(X(t+dt)) - F(X(t)) \\ &= F'(X(t)) \, dX + \frac{1}{2} F''(X(t)) \, dX^2 + \dots \\ &= F'(X(t)) \, dX + \frac{1}{2} F''(X(t)) \left[u \, dt + \sqrt{2D} \, dB \right]^2 + \dots \\ &= F'(X(t)) \, dX + DF''(X(t)) \, dB^2 + \mathcal{O}(dt^{3/2}); \end{aligned} \quad (1.28)$$

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hence, in a probabilistic sense, one has to leading order in dt

$$\begin{aligned} dF(X(t)) &= F'(X(t)) dX + D F''(X(t)) dt \\ &= [u F'(X(t)) + D F''(X(t))] dt + F'(X(t)) \sqrt{2D} dB(t). \end{aligned} \quad (1.29)$$

It is crucial to note that, due to the choice of the expansion point, the coefficient $F'(X)$ in front of $dB(t)$ is to be evaluated at $X(t)$. This convention is the so-called Ito integration rule. In particular, it is important to keep in mind that nonlinear transformations of Ito SDEs must feature second-order derivatives.

Numerical integration

Discretization dilemma To clarify the importance of discretization rules when dealing with SDEs, let us consider a simple generalization of Eq. (1.25), where drift u and diffusion coefficient D are position dependent. Adopting the Ito convention, the corresponding SDE reads

$$dX(t) = u(X) dt + \sqrt{2D(X)} * dB(t), \quad (1.30a)$$

where from now on the *-symbol signals that $D(X)$ is to be evaluated at $X(t)$. The simplest numerical integration procedure for Eq. (1.30a) is the stochastic Euler scheme

$$X(t + dt) = X(t) + u(X(t)) dt + \sqrt{2D(X(t))} \sqrt{dt} Z(t), \quad (1.30b)$$

$$Z(t) \sim \mathcal{N}(0, 1)$$

When you see an equation like (1.30a), then always ask which discretization rule has been adopted!

Ito vs. backward-Ito

Compare

$$dX(t) = u(X) dt + \sqrt{2D(X)} * dB(t), \quad (1.30a)$$

$$X(t + dt) = X(t) + u(X(t)) dt + \sqrt{2D(X(t))} \sqrt{dt} Z(t), \quad (1.30b)$$

with

the so-called backward Ito SDE with coefficients u_B and D_B , denoted by

$$dX(t) = u_B(X) dt + \sqrt{2D_B(X)} \bullet dB(t), \quad (1.31a)$$

is defined as the upper Riemann sum⁶

$$X(t + dt) = X(t) + u_B(X(t + dt)) dt + \sqrt{2D_B(X(t + dt))} \sqrt{dt} Z(t). \quad (1.31b)$$

do NOT give same results when $dt \rightarrow 0$

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For instance, the so-called backward Ito SDE with coefficients u_B and D_B , denoted by

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In particular

$$\begin{aligned} dF(X) &= F'(X) \bullet dX - D_B F''(X) dt \\ &= [u_B F'(X) - D_B F''(X)] dt + F'(X) \sqrt{2D_B} \bullet dB(t). \end{aligned} \quad (1.32)$$


Stratonovich SDE

Another discretization convention, that is popular in the physics literature is the Stratonovich-Fisk discretization, denoted by

$$dX(t) = u_S(X) dt + \sqrt{2D_S(X)} \circ dB(t), \quad (1.34a)$$

and defined as the mean value of lower and upper Riemann sum⁸

$$\begin{aligned} X(t+dt) = & X(t) + \frac{u_S(X(t)) + u_S(X(t+dt))}{2} dt + \\ & \frac{\sqrt{2D_S(X(t))} + \sqrt{2D_S(X(t+dt))}}{2} \sqrt{dt} Z(t). \end{aligned} \quad (1.34b)$$

From a numerical perspective, the non-anticipatory Ito scheme (1.30b) is advantageous for it allows to compute the new position directly from the previous one. For analytical calculations, the Stratonovich-Fisk scheme is somewhat preferable as it preserves the rules of ordinary differential calculus,⁹

$$dF(X) = F'(X) \circ dX(t) \quad (1.36)$$

each SDE formulation has advantages & disadvantages

Summary

Ito

$$dX(t) = u(X) dt + \sqrt{2D(X)} * dB(t), \quad (1.30a)$$

$$X(t + dt) = X(t) + u(X(t)) dt + \sqrt{2D(X(t))} \sqrt{dt} Z(t), \quad (1.30b)$$

backward-Ito

$$dX(t) = u_B(X) dt + \sqrt{2D_B(X)} \bullet dB(t)$$

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Stratonovich

$$dX(t) = u_S(X) dt + \sqrt{2D_S(X)} \circ dB(t)$$

$$X(t + dt) = X(t) + \frac{u_S(X(t)) + u_S(X(t + dt))}{2} dt + \frac{\sqrt{2D_S(X(t))} + \sqrt{2D_S(X(t + dt))}}{2} \sqrt{dt} Z(t)$$

1.2.2 Fokker-Planck equations

Since other types of SDEs can be transformed into an equivalent Ito SDE, we shall focus in this part on discussing how one can derive a Fokker-Planck equation (FPE) for the probability density function (PDF) $p(t, x)$ for a process $X(t)$ described by the Ito SDE

$$dX(t) = u(X) dt + \sqrt{2D(X)} * dB(t). \quad (1.37)$$

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$$\begin{aligned} \mathbb{E}[d[\delta(X - x)]] &= \mathbb{E}\left[(\partial_X \delta(X - x)) dX + D(X) \partial_X^2 \delta(X(t) - x) dt\right] \\ &= \mathbb{E}\left[(\partial_X \delta(X - x)) u(X) + D(X) \partial_X^2 \delta(X(t) - x)\right] dt. \end{aligned}$$

Here, we have used that $\mathbb{E}[g(X(t)) * dB] = 0$.

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Furthermore, by recalling that

$$\partial_X \delta(X - x) = -\partial_x \delta(X - x), \quad (1.40)$$

we may write

$$\begin{aligned} \mathbb{E}[d[\delta(X - x)]] &= \mathbb{E} \left[(-\partial_x \delta(X - x)) u(X) + D(X) \partial_x^2 \delta(X(t) - x) \right] dt \\ &= -\partial_x \mathbb{E}[\delta(X - x) u(X)] dt + \partial_x^2 \mathbb{E}[D(X) \delta(X(t) - x)] dt. \end{aligned}$$

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Using another property of the δ -function

$$f(y) \delta(y - x) = f(x) \delta(y - x) \quad (1.41)$$

we obtain

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Combining this with Eq. (1.39) yields the Fokker-Planck (or Smoluchowski) equation

$$\partial_t p = -\partial_x \{u(x) p - \partial_x [D(x)p]\}. \quad (1.42)$$

Ito-FPE

$$\partial_t p = -\partial_x \{u(x) p - \partial_x [D(x)p]\}$$

Backward-Ito FPE

For comparison, an analogous calculation for the backward-Ito SDE

$$dX(t) = u_B(X) dt + \sqrt{2D_B(X)} \bullet dB(t), \quad (1.43)$$

gives

$$\partial_t p = -\partial_x [u_B(x) p - D_B(x) \partial_x p]. \quad (1.44)$$

Compared with the Ito FPE (1.42), the diffusion coefficient D_B now enters in front of the gradient $\partial_x p$. Note, however, that the two different FPEs coincide if one identifies the coefficients as in Eq. (1.33):

$$u = u_B + \partial_x D_B, \quad D = D_B$$